

CSC 144V: Autonomous Vehicles

Lecture 8

Control Exercises Solution

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Exercise: PID by hand

Lateral PID with $\Delta t = 1$: **steer** = $-K_p \cdot e - K_d \cdot (e - e_{\text{prev}}) - K_i \cdot \Sigma e$. $K_p = 0.2$, $K_d = 1.0$, $K_i = 0.004$. Cross-track error: $e_0 = 1.0$ (initial), $e_1 = 0.7$, $e_2 = 0.3$, $e_3 = -0.1$. Σe includes the current error.

(a) Fill in the table.

(b) What would a P-only controller ($K_d = K_i = 0$) output at $t = 1, 2, 3$? What does the D term do?

t	e	$e - e_{\text{prev}}$	Σe	steer
1	0.7			
2	0.3			
3	-0.1			

Exercise Solution: PID by hand

- $t = 1$: $-(0.2 \cdot 0.7 + 1.0 \cdot (-0.3) + 0.004 \cdot 0.7) = -(0.14 - 0.30 + 0.0028) = \mathbf{0.157}$; $t = 2$: $-(0.06 - 0.40 + 0.004) = \mathbf{0.336}$; $t = 3$: $-(-0.02 - 0.40 + 0.0036) = \mathbf{0.416}$.
- P-only: -0.14 , -0.06 , $\mathbf{+0.02}$: it steers back, but reaches zero error with full steering authority $\rightarrow$ overshoot (the sign flips once $e < 0$).
- The D term opposes the fast decrease of e : it **damps** the response and reduces overshoot (here it even reverses the steering sign at $t = 1$).

t	e	$e - e_{\text{prev}}$	Σe	steer
1	0.7	-0.3	0.7	0.157
2	0.3	-0.4	1.0	0.336
3	-0.1	-0.4	0.9	0.416

Exercise: Effects of PID gains

Fill in the effect of **increasing** each gain on the step-response metrics (↑ increase, ↓ decrease, – little change).

(b) A car oscillates around the lane center with large overshoot. Which gain do you raise?

(c) On a constant-curvature track the car keeps a persistent offset (steady-state CTE). Which gain removes it?

Gain	Rise time	Overshoot	Settling time	Steady-state error
K_p				
K_i				
K_d				

Exercise Solution: Effects of PID gains

- (b) **Kd** (derivative): it damps the oscillation and lowers overshoot.
- (c) **Ki** (integral): it accumulates the persistent error until it is eliminated (at the price of more overshoot).
- K_p alone trades a faster response for more overshoot; a P-only controller leaves a steady-state error.

Gain	Rise time	Overshoot	Settling time	Steady-state error
K_p	↓	↑	—	↓
K_i	↓	↑	↑	eliminated
K_d	—	↓	↓	—

Exercise: Kinematic bicycle model

Kinematic bicycle model (rear axle): $x \leftarrow x + v \cos\theta \Delta t$, $y \leftarrow y + v \sin\theta \Delta t$, $\theta \leftarrow \theta + (v/L) \tan\delta \Delta t$. $L = 2.5$ m, $v = 10$ m/s, constant steering $\delta = 0.1$ rad, $\Delta t = 0.1$ s, start $(0, 0, \theta = 0)$.

- (a) Compute the state after steps 1 and 2.
- (b) Turning radius $R = L/\tan\delta$ and lateral acceleration v^2/R ?
- (c) For a comfort limit of 2 m/s^2 at $v = 10$ m/s, what is the maximum steering angle?

Step	x (m)	y (m)	θ (rad)
0	0	0	0
1			
2			

Exercise Solution: Kinematic bicycle model

- (a) $\theta_1 = (10/2.5) \cdot \tan 0.1 \cdot 0.1 = 0.0401$; $x_1 = 1.000$, $y_1 = 0$.
Step 2: $x_2 = 1 + \cos(0.0401) = 1.999$, $y_2 = \sin(0.0401) = 0.0401$, $\theta_2 = 0.0803$.
- (b) $R = 2.5/\tan 0.1 \approx \mathbf{24.9 \text{ m}}$; $v^2/R = 100/24.9 \approx \mathbf{4.0 \text{ m/s}^2}$.
- (c) $R_{\min} = v^2/a = 100/2 = 50 \text{ m}$; $\delta_{\max} = \text{atan}(L/R) = \text{atan}(0.05) \approx \mathbf{0.05 \text{ rad} \approx 2.9^\circ}$.

Step	x (m)	y (m)	θ (rad)
0	0	0	0
1	1.000	0.000	0.0401
2	1.999	0.0401	0.0803

Exercise: Pure pursuit and Stanley

Wheelbase $L = 2.5$ m. **Pure pursuit:** $\delta = \text{atan}(2 L \sin\alpha / L_d)$, α = angle to the look-ahead point. **Stanley:** $\delta = \psi + \text{atan}(k e / v)$.

(a) Pure pursuit with $\alpha = 20^\circ$, $L_d = 5$ m: δ ?

(b) Same α with $L_d = 2$ m: δ ?

(c) Stanley with heading error $\psi = 0.1$ rad, cross-track error $e = 0.5$ m, $k = 1.5$, $v = 6$ m/s: δ ?

(d) Peak error in a 20 m radius corner ($v = 5$ m/s): $L_d = 2$ m $\rightarrow$ 0.02 m, 5 m $\rightarrow$ 0.13 m, 10 m $\rightarrow$ 0.55 m. Trade-off of the look-ahead distance?

Exercise Solution: Pure pursuit and Stanley

- (a) $2 \cdot 2.5 \cdot \sin 20^\circ / 5 = 0.342 \rightarrow \delta = \text{atan}(0.342) = \mathbf{0.330 \text{ rad} \approx 18.9^\circ}$.
- (b) $5 \cdot 0.342 / 2 = 0.855 \rightarrow \delta = \mathbf{0.708 \text{ rad} \approx 40.6^\circ}$: a short look-ahead steers much harder.
- (c) $\text{atan}(1.5 \cdot 0.5/6) = \text{atan}(0.125) = 0.124$; $\delta = 0.1 + 0.124 = \mathbf{0.224 \text{ rad} \approx 12.9^\circ}$ (both terms steer toward the path).
- (d) A large L_d is smooth and stable but **cuts corners** (peak error grows $0.02 \rightarrow 0.55 \text{ m}$); a small L_d tracks the path closely but steers aggressively and can oscillate at speed.

Exercise: LQR trade-offs

Scalar system $\dot{x} = a x + b u$ with $a = 1$ (unstable), $b = 1$; cost $\int (q x^2 + r u^2) dt$. The Riccati equation $2ap - (b^2/r)p^2 + q = 0$ gives p ; the gain is $K = b p / r$, $u = -K x$, and the closed-loop pole is $a - bK$.

Fill in the table. What do larger q and larger r do?

q	r	p	K	Pole
1	1			
4	1			
1	4			

Exercise Solution: LQR trade-offs

- $q = r = 1$: $-p^2 + 2p + 1 = 0 \rightarrow p = 1 + \sqrt{2} = 2.414$, $K = 2.414$, pole = $1 - 2.414 = -1.414$.
- $q = 4, r = 1$: $p = 1 + \sqrt{5} = 3.236$, $K = 3.236$, pole -2.236 (faster).
- $q = 1, r = 4$: $p^2 - 8p - 4 = 0 \rightarrow p = 8.472$, $K = 8.472/4 = 2.118$, pole -1.118 (slower).
- Larger **q** penalizes state error more: more aggressive. Larger **r** makes control expensive: gentler. No pole locations are chosen by hand.

q	r	p	K	Pole
1	1	2.414	2.414	-1.414
4	1	3.236	3.236	-2.236
1	4	8.472	2.118	-1.118