Embedded Systems with ARM Cortex-M Microcontrollers in Assembly Language and C

Chapter 13
Fixed-point Numbers

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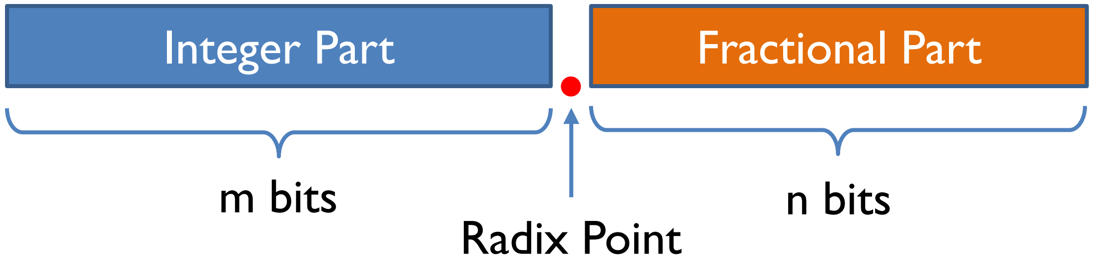
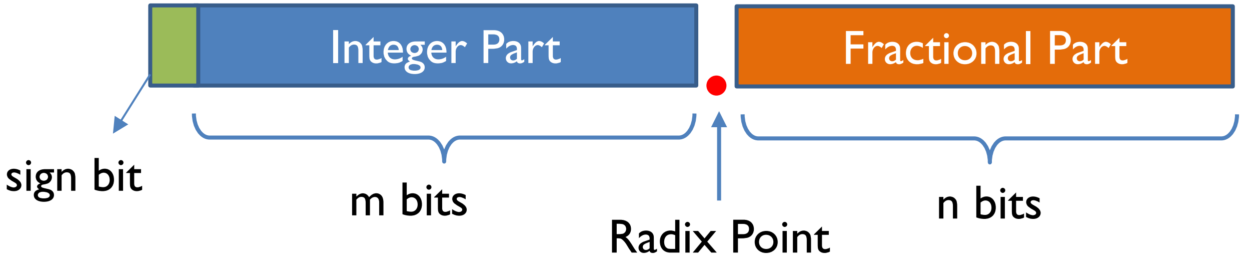
Acknowledgement: Lecture slides based on Embedded Systems with ARM Cortex-M Microcontrollers in Assembly Language and C, University of Maine https://web.eece.maine.edu/~zhu/book/

Fixed-point Format: Q Notation

UQm.n for unsigned fixed-point

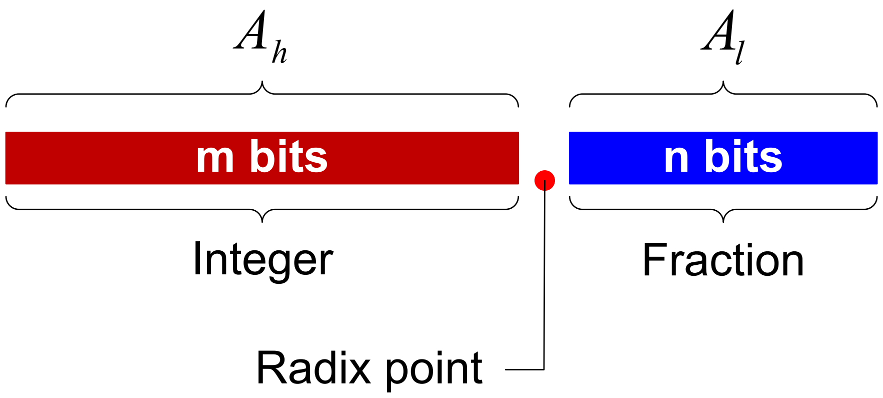


Qm.n for signed fixed-point

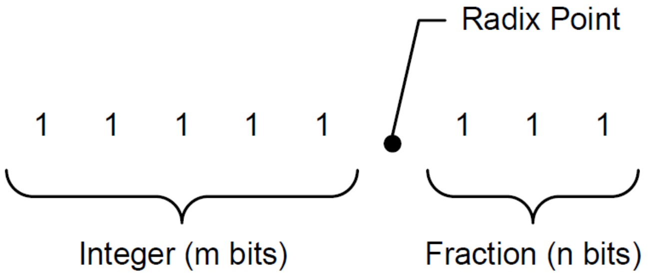


m + n + 1 bits

Q Notation: **UQm.n**

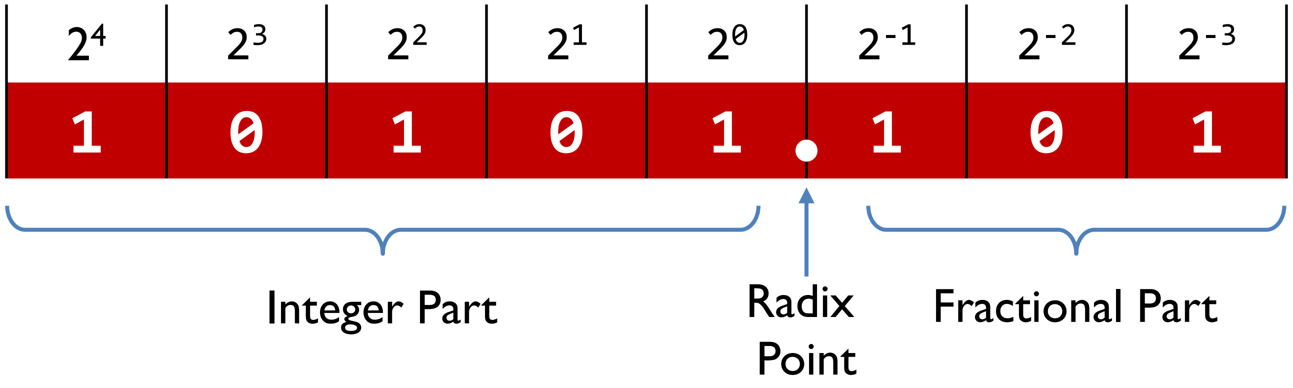


$$f = A_h + A_l \times 2^{-n}$$



$$\begin{aligned} 10101.101_2 &= A_h + A_l \times 2^{-3} \\ &= 21 + 5 \times 2^{-3} \\ &= 21.625 \end{aligned}$$

Converting Fixed-point UQ5.3 to Float



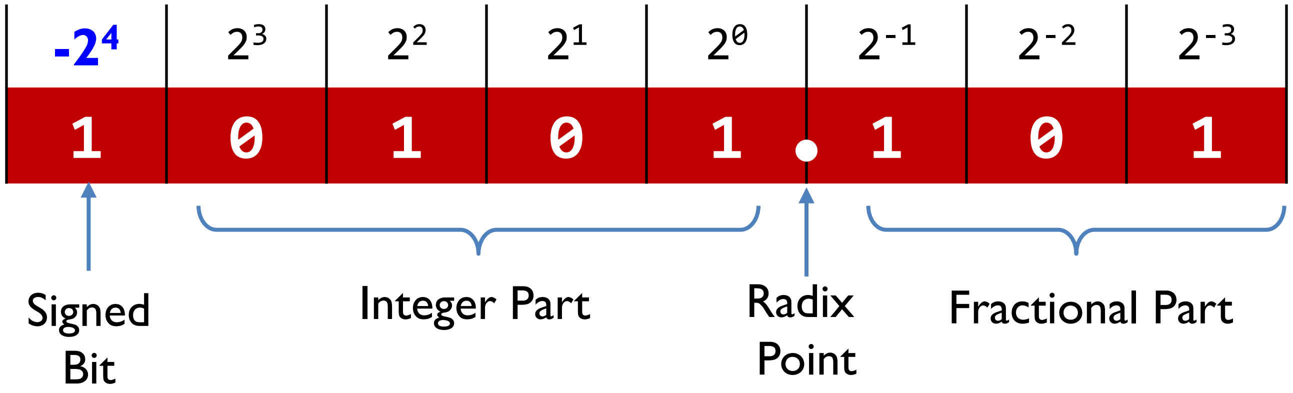
$$\mathbf{10101.101}_2$$

$$= \mathbf{1} \times 2^4 + \mathbf{0} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{0} \times 2^1 + \mathbf{1} \times 2^0 + \mathbf{1} \times 2^{-1} + \mathbf{0} \times 2^{-2} + \mathbf{1} \times 2^{-3}$$

$$= 21.625$$

$$\mathbf{10101.101}_2 = \frac{10101101_2}{2^3} = \frac{173}{8} = 21.625$$

Converting Fixed-point Q4.3 to Float



$\mathbf{10101.101}_2$

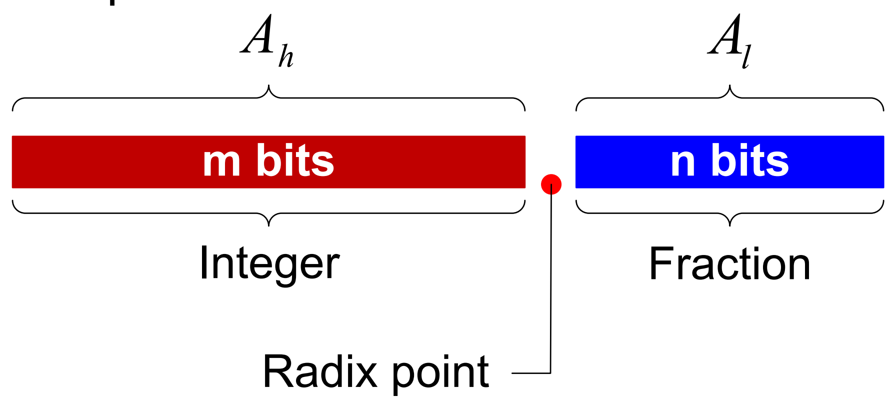
$= \mathbf{1} \times (-\mathbf{2^4}) + \mathbf{0} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{0} \times 2^1 + \mathbf{1} \times 2^0 + \mathbf{1} \times 2^{-1} + \mathbf{0} \times 2^{-2}$
$+ \mathbf{1} \times 2^{-3}$

$= -10.375$

$$\mathbf{10101.101}_2 = \frac{10101101_2}{2^3} = \frac{-83}{8} = -10.375$$

U5.3 vs. Q4.3

Unsigned Fixed-point Representation **Um.n**



$$f = A_h + A_l \times 2^{-n}$$

$$10101.101_2 = A_h + A_l \times 2^{-3}$$

$$= 21 + 5 \times 2^{-3}$$

$$= 21.625$$

Signed Fixed-point Representation **Qm.n**

$$f = A_h + A_l \times 2^{-n}$$

$$10101.101_2 = A_h + A_l \times 2^{-3}$$

$$= -11 + 5 \times 2^{-3}$$

$$= -10.375$$

Two Ways of Calculating Two's Complement (integer)

- Convert 10101 into decimal with Two's Complement notation'
- Method 1, invert bits and add 1:
  - 01010 + 1 = 01011 = 11 in decimal, hence 10101 = -11 in decimal
- Method 2, calculate directly:
  - $\mathbf{1} \times \left(-\mathbf{2}^4\right) + \mathbf{0} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{0} \times 2^1 + \mathbf{1} \times 2^0 = -16 + 4 + 1 = -11$ in decimal
- The two definitions are mathematically identical (proof omitted)

Convert Float to Fixed-point UQ4.12

UQm.n for unsigned fixed-point

$$Representation = round(float \times 2^n)$$

Example: Convert $f = 3.141593$ to UQ4.12 (4 bits integer, 12 bits fraction)

- Calculate $f \times 2^{12} = 12867.964928$
- Round the result to nearest integer, $round(12867.964928) = 12868$
- Convert the integer to binary: 12868 = **0011_0010_0100_0100**$_2$
- Organize into UQ4.12: **0011.0010_0100_0100**$_2$
- Final result in hex: **0x3244**
- Error = reconstructed_value − true_value = $\frac{12868}{2^{12}} - f = 8.5625 \times 10^{-6}$

Convert Float to Fixed-point Q3.12

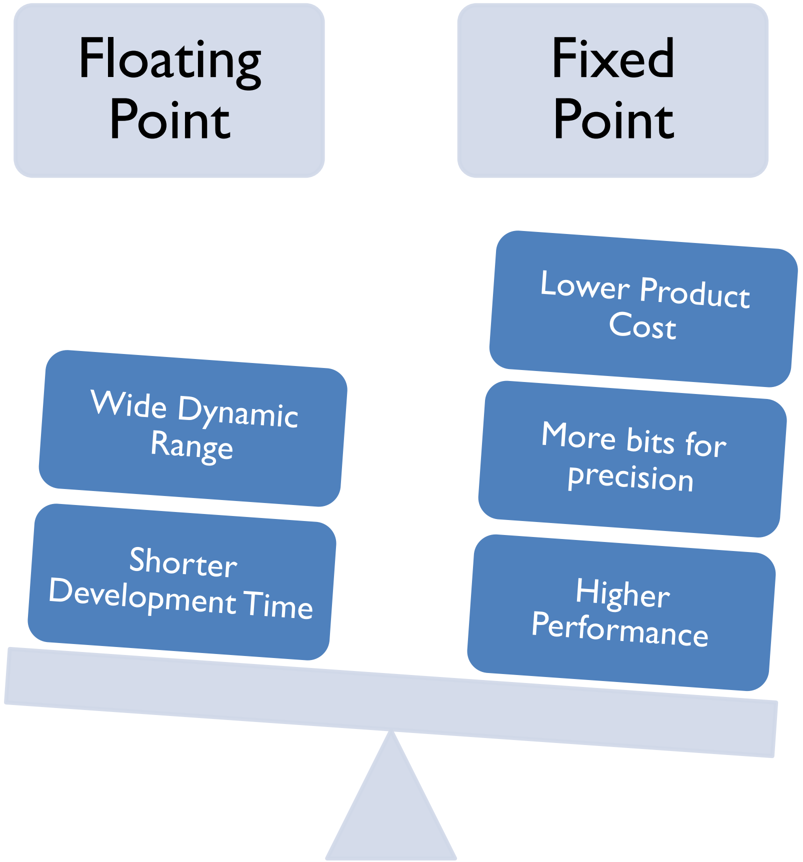
Qm.n for signed fixed-point

$$Representation = round(float \times 2^n)$$

Example: Convert $f = -3.141593$ to Q3.12 (1 sign bit, 3 bits integer, 12 bits fraction)

- Calculate $f \times 2^{12} = -12867.964928$
- Round the result to an integer, $round(-12867.964928) = -12868$
- Find the 16-bit two's complement: **`1100_1101_1011_1100`**$_2$
  - $12868 = 0x3244 = 0011\ 0010\ 0100\ 0100_2$
  - Invert bits and add 1 $\rightarrow -12868 = 1100\ 1101\ 1011\ 1100_2$
- Organize into Q3.12: **`1100.1101_1011_1100`**$_2$
- Final result in hex: **`0xCDBC`**
- Error = reconstructed_value − true_value = $-\frac{12868}{2^{12}} - f = 8.5625 \times 10^{-6}$

Why use fixed-point?



Embedded Systems with ARM Cortex-M Microcontrollers in Assembly Language and C

Chapter 13
Floating-point Numbers

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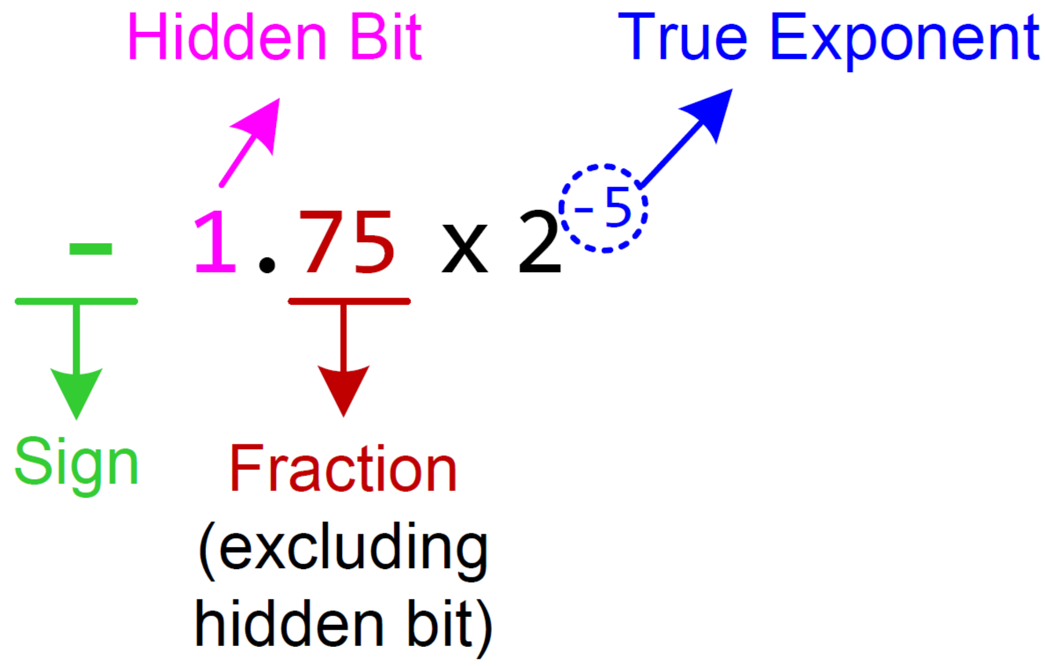
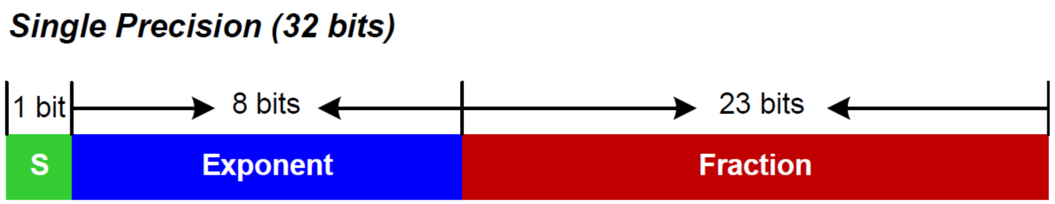
Normalization



We can convert 10.746×2^6 to normalized format as follows:

$$10.746 \times 2^6 = \frac{10.746}{8} \times 8 \times 2^6 = 1.34325 \times 2^9$$

IEEE Standard 754 (Signed Floating Point)



IEEE 754 value:

$$(-1)^{S} \times (1 + Fraction) \times 2^{Exponent - Bias}$$

where Bias = 2^7 - 1 = **127** for single precision FP32

Bias = 2^{10} - 1 = **1023** for double precision FP64

Hidden bit refer to the implicit leading 1 in (1 + fraction)

Fraction is also called significand or mantissa

Why Bias?

- FP32's exponent field has 8 bits. So it can represent integers from 0 to 255. But real exponents in normalized floating-point numbers range roughly from –126 to +127. To store both negative and positive exponents in that 0–255 range, IEEE-754 adds a bias, shifting all values upward so they become non-negative.
- For an exponent field of n bits, the bias is $2^{(n-1)}$- 1.
- For FP32, the bias is chosen as $2^{(8-1)}$- 1 = 127, so half the range is allocated for negative exponents, and half for positive.
  - Actual Exponent=Stored Exponent -127
  - Stored exponent 0 → actual exponent = -127 (used for special/subnormal cases)
  - Stored exponent 127 → actual exponent = 0
  - Stored exponent 255 → reserved for special values (like infinity and NaN)

Decoding 0xC1FF0000
into a floating-point number

- Binary 11000001111111110000000000000000
- Sign = 1
- Exponent = $10000011_2 = 131$
- Fraction = $0.1111111_2 = 1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4} + 1 \times 2^{-5} + 1 \times 2^{-6} + 1 \times 2^{-7} = 0.9921875$
- $$\begin{aligned} f &= (-1)^S \times (1 + Fraction) \times 2^{Exponent-127} \\ &= (-1)^1 \times (1 + 0.9921875) \times 2^{131-127} \\ &= -1 \times 1.9921875 \times 2^4 \\ &= -31.875 \end{aligned}$$
- If Exponent = $10000101_2 = 133$, then $f = -1 \times 1.9921875 \times 2^6 = -127.5$
- If Exponent = $10000110_2 = 134$, then $f = -1 \times 1.9921875 \times 2^7 = -255$

$$\underset{\text{Sign Bit}}{1},\underbrace{\overbrace{10000011}^{\text{8 bits}}}_{\text{Exponent}},\underbrace{\overbrace{11111110000000000000000}^{\text{23 bits}}}_{\text{Fraction}}$$

Decoding 0x40920000 into a floating-point number

- Binary 01000000100100000000000000000000
- Sign = 0
- Exponent = $10000001_2 = 129$
- Fraction = $0.001_2 = 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} = 0.125$
- $f = (-1)^S \times (1 + Fraction) \times 2^{Exponent-127}$

$$= (-1)^0 \times (1 + 0.125) \times 2^{129-127}$$
$$= -1 \times 1.125 \times 2^2$$
$$= 4.5$$

- How Floating Point Numbers Work (in 7 minutes!)
  - https://www.youtube.com/watch?v=W_Knvo9NuJY

Decoding 0x3F800000 into a floating-point number

- Binary 00111111100000000000000000000000
- Sign = 0
- Exponent = $01111111_2 = 127$
- Fraction = 0
- $f = (-1)^S \times (1 + Fraction) \times 2^{Exponent-127}$
$$= (-1)^0 \times (1 + 0) \times 2^{127-127}$$
$$= -1 \times 1 \times 2^0$$
$$= 1.0$$
- How Floating Point Numbers Work (in 7 minutes!)
  - https://www.youtube.com/watch?v=W_Knvo9NuJY

Decoding `0x41680000` into a floating-point number

- Binary `01000001011010000000000000000000`
- Sign = `0`
- Exponent = $10000010_2 = 130$
- Fraction = $0.1101_2 = 1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-4} = 0.8125$
- $f = (-1)^S \times (1 + Fraction) \times 2^{Exponent-127}$

$$
\begin{aligned}
&= (-1)^0 \times (1 + 0.8125) \times 2^{130-127} \\
&= 1 \times 1.8125 \times 2^3 \\
&= 14.5
\end{aligned}
$$

Encoding 14.5 into IEEE Std 754 Single-Precision

- Normalization:
- $2^3 < 14.5 < 2^4, \frac{14.5}{2^3} = 1.8125$
- Hence $14.5 = 1.8125 \times 2^3 = (1 + 0.8125) \times 2^3$
- Conversion:
- $Sign = 0$
- $Exponent = 3 + 127 = 130 = 1000010_2$
- $Fraction = 0.1101_2$ (multiply by 2 repeatedly)
- Assume $Fraction = b1 \times 2^{-1} + b2 \times 2^{-2} + b3 \times 2^{-3} + b4 \times 2^{-4} + \cdots$
  - $0.8125 \times 2 = 1.625 = 1 + 0.625$ => b1 = 1
  - $0.625 \times 2 = 1.25 = 1 + 0.25$ => b2 = 1
  - $0.25 \times 2 = 0.5 = 0 + 0.5$ => b3 = 0
  - $0.5 \times 2 = 1$ => b4 = 1
- `14.5` = `01000001011010000000000000000000` in binary or `0x41680000` in hex

Encoding 1.3 into IEEE Std 754 Single-Precision

- Normalization:
- $2^0 < 1.3 < 2^1, \frac{1.3}{2^0} = 1.3$
- Hence $1.3 = \ (1 + 0.3) \times 2^0$
- Conversion:
- $Sign = 0$
- $Exponent = 0 + 127 = 127 = 01111111_2$
- $Fraction = 0.01001100110011\ \ldots_2$ (multiply by 2 repeatedly)
- Assume $Fraction = b1 \times 2^{-1} + b2 \times 2^{-2} + b3 \times 2^{-3} + b4 \times 2^{-4} + \cdots$
  - $0.3 \times 2 = 0.6$ => b1 = 0
  - $0.6 \times 2 = 1.2$ => b2 = 1
  - $0.2 \times 2 = 0.4$ => b3 = 0
  - $0.4 \times 2 = 0.8$ => b4 = 0
  - $0.8 \times 2 = 1.6$ => b5 = 1
  - $0.6 \times 2 = 1.2$ => b6 = 1

  Repeats infinitely
- 14.5 = 00111111101001100110011001100110 in binary or 0x3FA66666 in hex

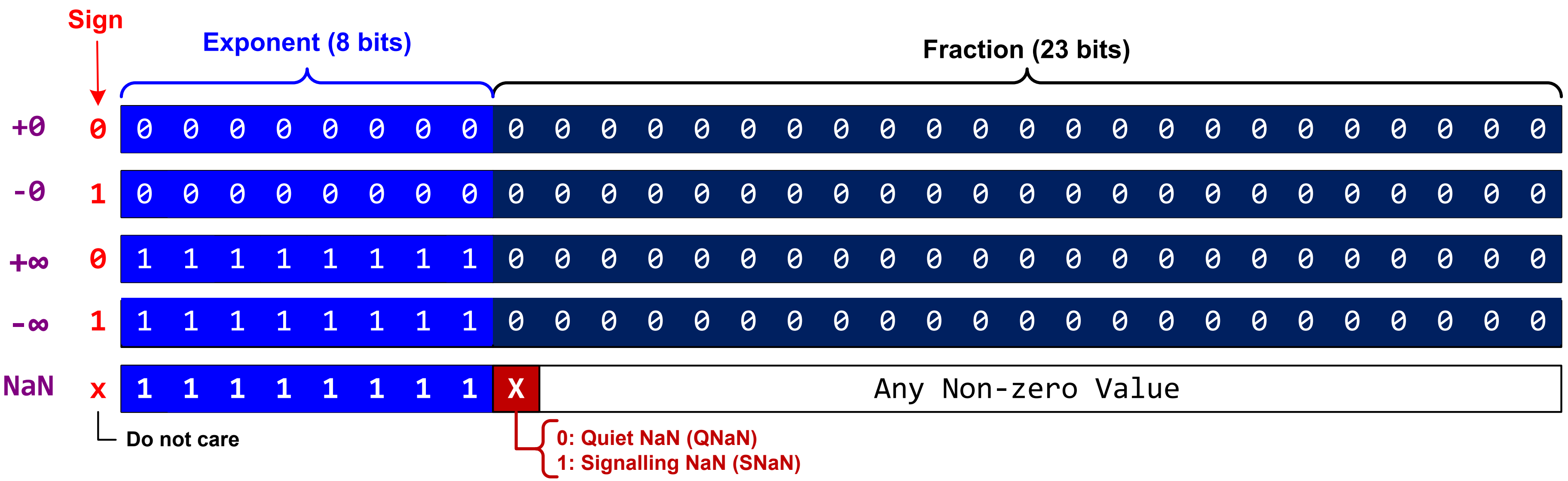
Decoding `0x3FA66666` into a floating-point number

- Binary `00111111010011001100110011001100110`
- Sign = `0`
- Exponent = $01111111_2 = 127$
- Fraction = 0.29999995 (calculation process skipped)
- $f = (-1)^S \times (1 + Fraction) \times 2^{Exponent-127}$
  $= (-1)^0 \times (1 + 0.29999995) \times 2^{127-127}$
  $= 1.29999995$
- Error: $1.3 - 1.29999995 = 5 \times 10^{-8}$

Why Is This Happening?! Floating Point Approximation4
https://www.youtube.com/watch?v=2gIxbTn7GSc

Special Values

- Exponents 00000000 and 111111111 are reserved.



- Example of Not-A-Number (NaN)
  - `log(-10.0), sqrt(-1.0), 0.0/0.0,` $-\infty + \infty$,

Subnormalized Float Number

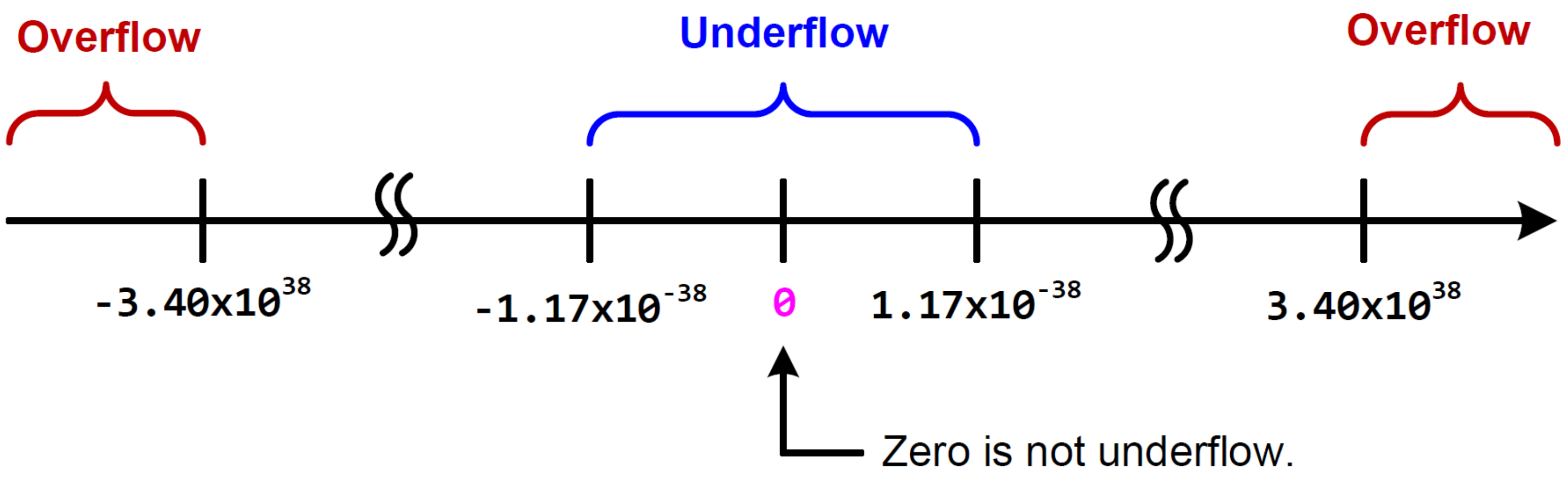
- To represent numbers between 0 and the minimum positive number that the normalized format can represent.
- **Normalized Format**

$$(-1)^S \times (\mathbf{1} + Fraction) \times 2^{\mathrm{Exponent}-127}$$

- **Sub-normalized** Format

$$(-1)^S \times Fraction \times 2^{-126}$$

Overflow and Underflow



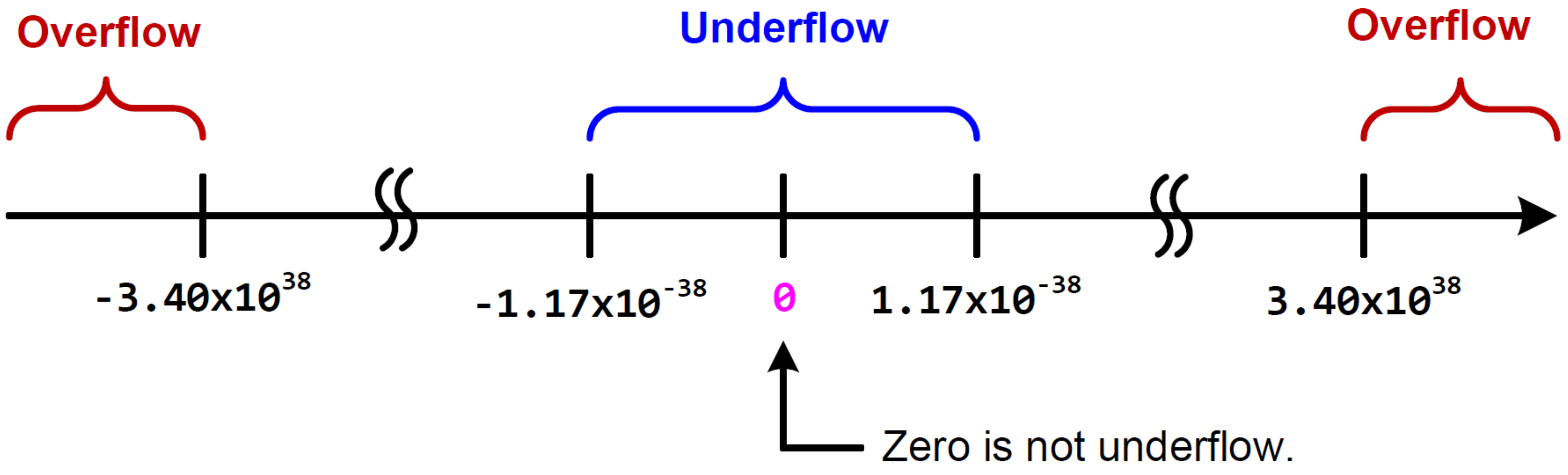
Smallest Positive Normal Number: `0 00000001 00000000000000000000000`

$$(-1)^0 \times (1+0) \times 2^{1-127} = 2^{-126} \approx 1.18 \times 10^{-38}$$

Smallest Positive Subnormal Number: `0 00000000 00000000000000000000001`

$$(-1)^0 \times (0+2^{-23}) \times 2^{1-127} = 2^{-149} \approx 1.40 \times 10^{-45}$$

Overflow and Underflow

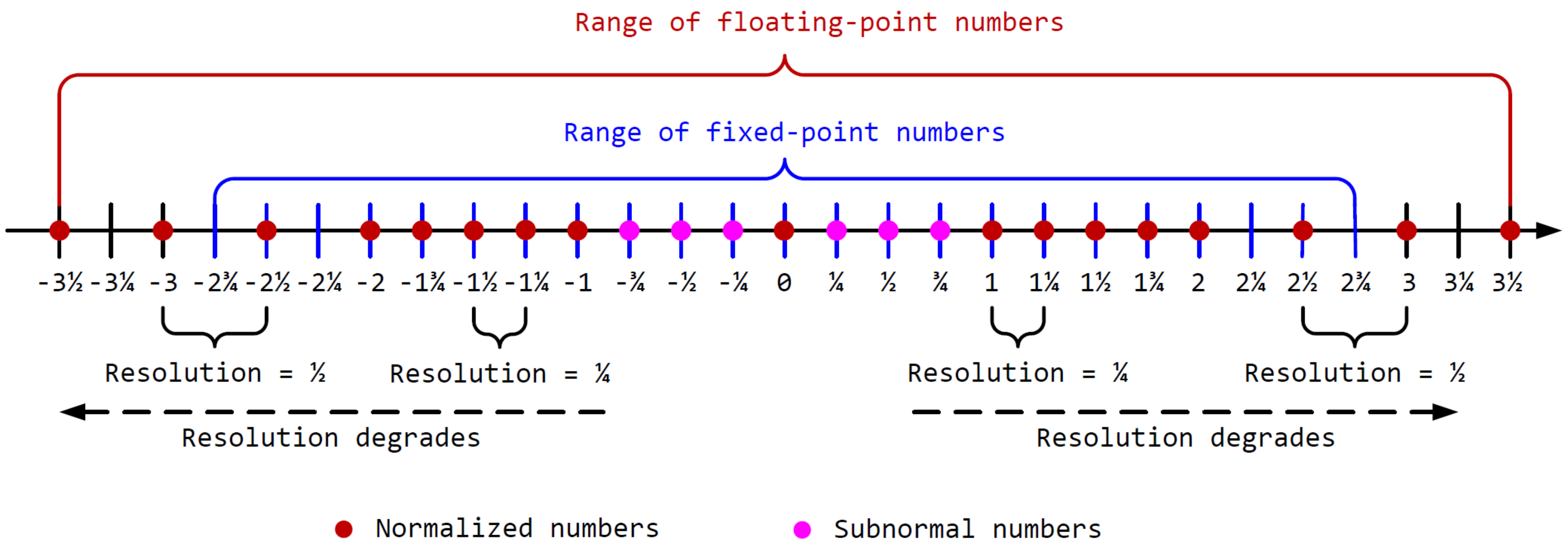


To find the largest representable number:
Exponent = largest possible **finite** value = 254 (since 255 is reserved for infinity and NaN)
Mantissa = all 1s $\rightarrow 1.11111111111111111111111_2 = 1 + (1 - 2^{-23})$
Numbers farthest from zero:

$$(-1)^S \times \left(1 + (1 - 2^{-23})\right) \times 2^{254-127} = \pm(2^{128} - 2^{104}) \approx \pm 3.40 \times 10^{38}$$

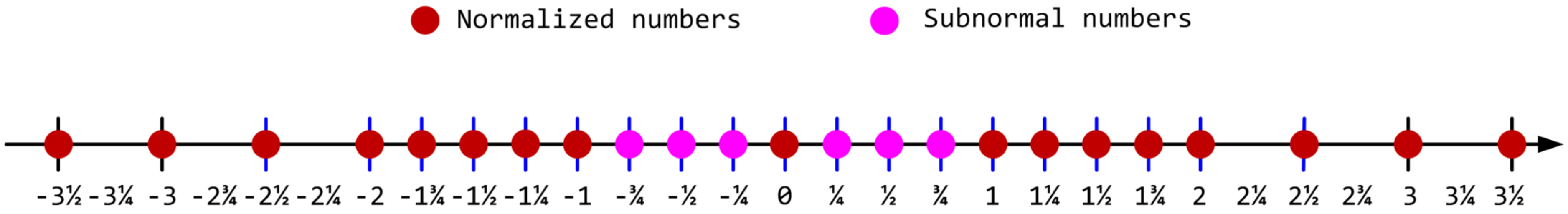
Resolution

- Given a hypothetical five-bit floating-point system (similar to IEEE 754).
  - the sign bit, an exponent (2 bits), and a fraction (2 bits)



Tradeoff between Range and Precision

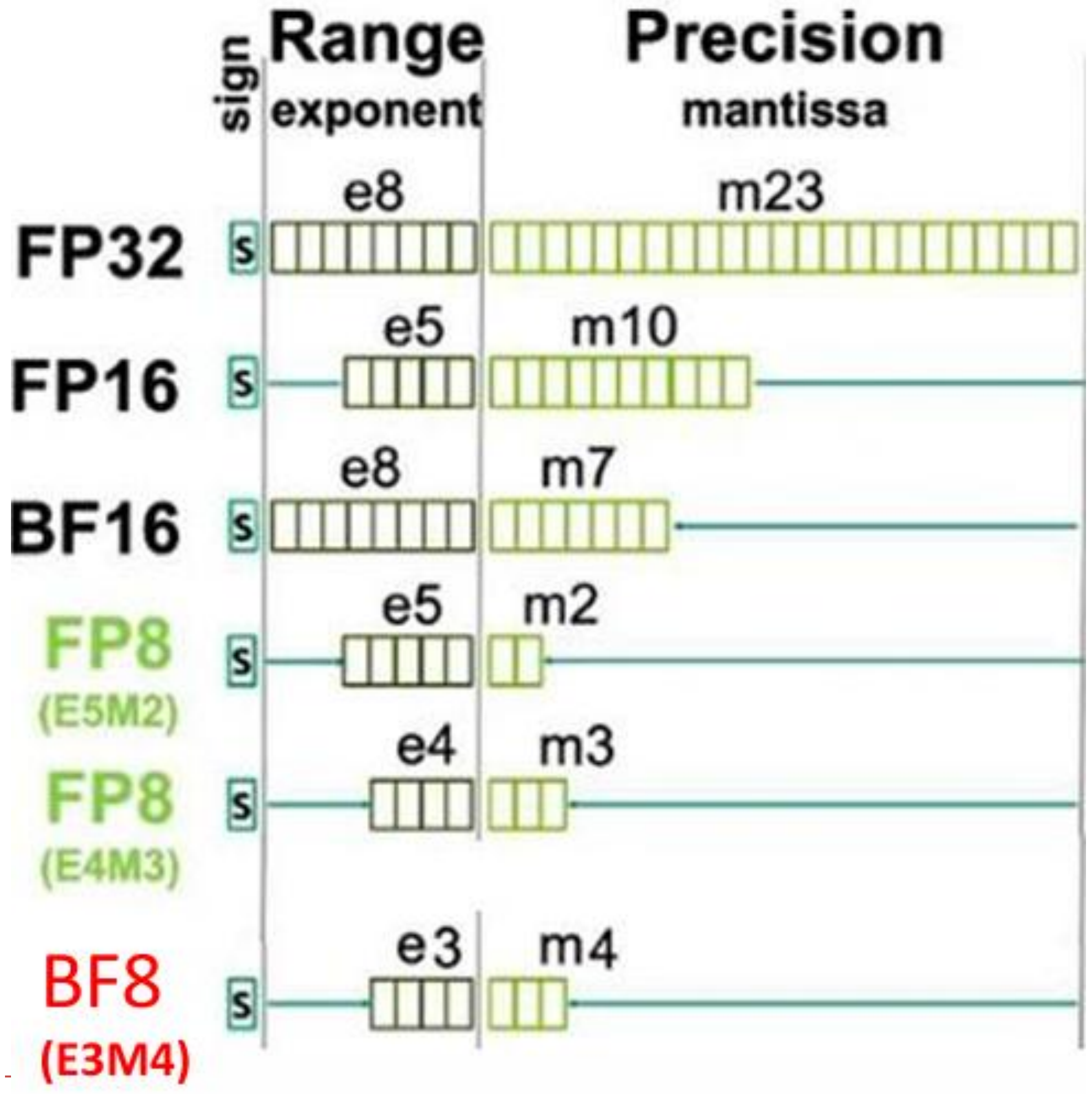
A simplified 5-bit floating-point (IEEE 754 style)



- Floating-Point
  - Resolution: difference between two neighbor numbers
  - Precision decreases as the magnitude increases
- Fixed-Point
  - Numbers are evenly distributed among the representable range
  - Precision is fixed

FP formats used in AI and machine learning

- bf16 (bfloat16) and bf8 (bfloat8) are floating point formats used in AI and machine learning for efficient computation with lower precision while retaining useful range.
- bf8 is cutting edge and experimental for very efficient AI deployment where accuracy can be slightly sacrificed for speed and lower memory footprints. bf16 is established as a good practical balance for many ML tasks



References

- How Floating Point Numbers Work (in 7 minutes!)
  - https://www.youtube.com/watch?v=W_Knvo9NuJY
- HOW TO: Convert IEEE-754 Single-Precision Binary to Decimal, Steven Petryk
  - https://www.youtube.com/watch?v=4DfXdJdaNYs
- Why Is This Happening?! Floating Point Approximation
  - https://www.youtube.com/watch?v=2gIxbTn7GSc
- IEEE 754 Binary to Float Conversion
  - https://www.youtube.com/watch?v=9k5rdPUzij8&list=PL-ftFcielQtGxBUfzkbz9tWIRZb-rlJ2p&index=2
- IEEE 754 Float to Binary Conversion
  - https://www.youtube.com/watch?v=9k5rdPUzij8&list=PL-ftFcielQtGxBUfzkbz9tWIRZb-rlJ2p&index=2