Lecture 13
Sorting Algorithm

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Lecture Goals

- We introduce simple sorting algorithms, incl. Bubble Sort, Selection Sort, and Insertion Sort, with high time complexity.
- We introduce *binary heap* for priority queue data abstract, which leads to an efficient sorting algorithm known as Heap Sort.
- We introduce the Quick Sort algorithm and analyze its performance.
- We introduce the Merge Sort algorithm and analyze its performance.

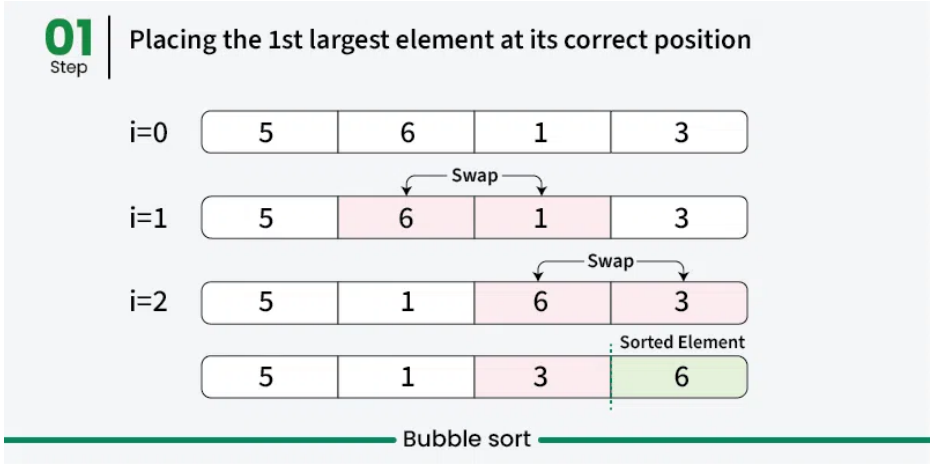
Bubble Sort

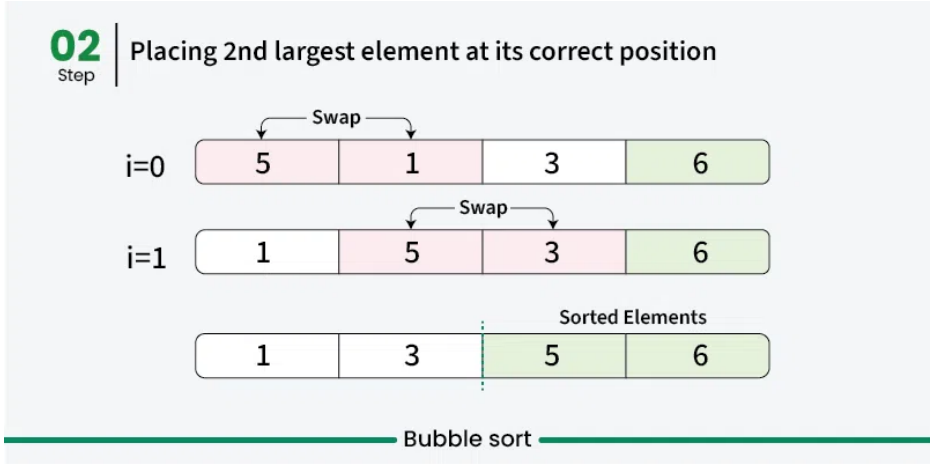
- Bubble Sort works by repeatedly swapping adjacent elements if they are in the wrong order.
  - We sort the array using multiple passes. After the first pass, the maximum element goes to end (its correct position). Same way, after second pass, the second largest element goes to second last position and so on.
  - In every pass, we process only those elements that have already not moved to correct position. After k passes, the largest k elements must have been moved to the last k positions.
- Time complexity: $O(n^2)$
- Bubble Sort | GeeksforGeeks
  - https://www.geeksforgeeks.org/bubble-sort-algorithm/
  - https://www.geeksforgeeks.org/time-and-space-complexity-analysis-of-bubble-sort/
  - https://www.youtube.com/watch?v=nmhjrI-aW5o

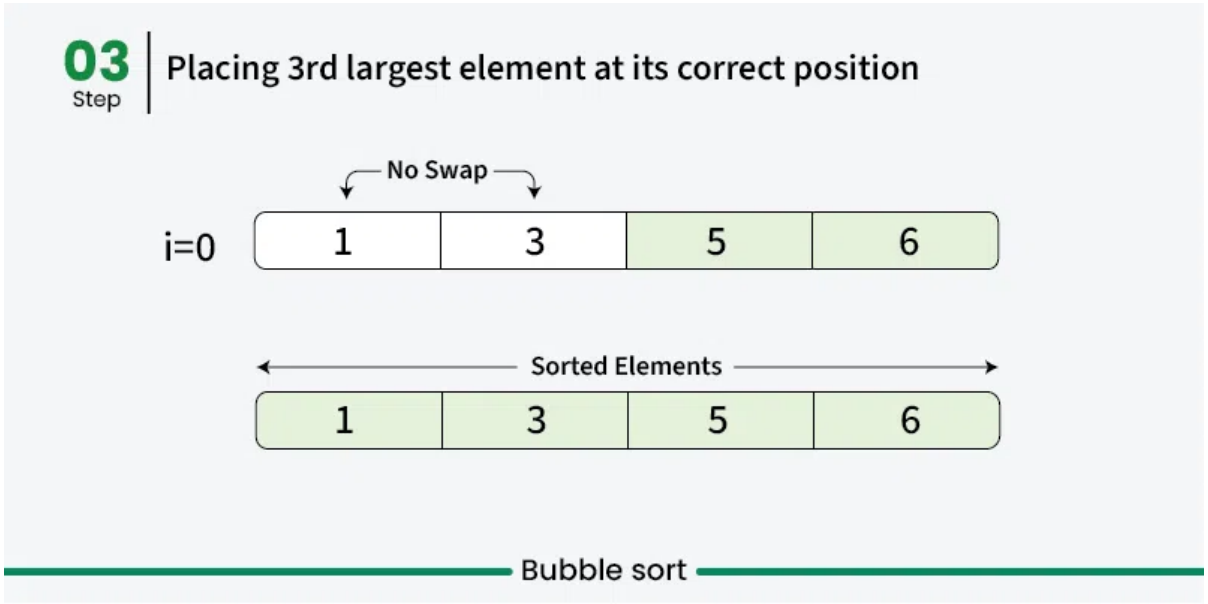
Bubble Sort Time Complexity

- The worst-case condition for bubble sort occurs when elements of the array are arranged in decreasing order. In the worst case, the total number of iterations or passes required to sort a given array is (n-1). where n is the number of elements present in the array.
- ***Pass 1:***
  Number of comparisons = (n-1), Number of swaps = (n-1)
- ***Pass 2:***
  Number of comparisons = (n-2), Number of swaps = (n-2)
- ***…***
- ***Pass n-1:***
  Number of comparisons = 1
  Number of swaps = 1
- *Total number of comparison required to sort the array*
  = (n-1) + (n-2) + . . . 2 + 1
  = (n-1)(n-1+1)/2 { by using sum of n natural Number formula }*
  *= **(n*(n-1)) / 2***
- *In worst case, **Total number of swaps = Total number of comparison***
  *Total number of comparison (Worst case) = **n(n-1)/2***
  *Total number of swaps (Worst case) = **n(n-1)/2***
- *So worst case time complexity is **$O(n^2)$***

Bubble Sort Example



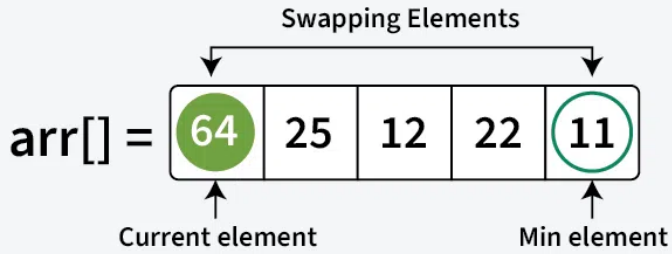




Selection Sort

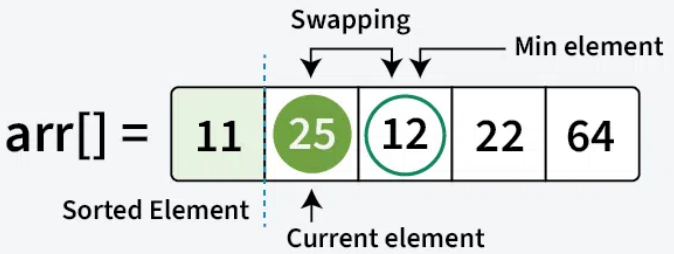
- Selection Sort works by repeatedly selecting the smallest element from the unsorted portion and swapping it with the first unsorted element. This process continues until the entire array is sorted.
  - First we find the smallest element and swap it with the first element. This way we get the smallest element at its correct position.
  - Then we find the smallest among remaining elements (or second smallest) and move it to its correct position by swapping.
  - Keep going until all elements are sorted.
- Time complexity: $O(n^2)$, as there are two nested loops:
  - Outer loop to select each element one by one with O(n) complexity
  - Inner loop to compare that element with every other element with O(n) complexity
- Selection Sort | GeeksforGeeks
  - https://www.geeksforgeeks.org/selection-sort-algorithm-2/
  - https://www.geeksforgeeks.org/time-and-space-complexity-analysis-of-selection-sort/
  - https://www.youtube.com/watch?v=xWBP4lzkoyM

01 Step Start from the first element at index 0, find the smallest element in the rest of the array which is unsorted, and swap (11) with current element(64).



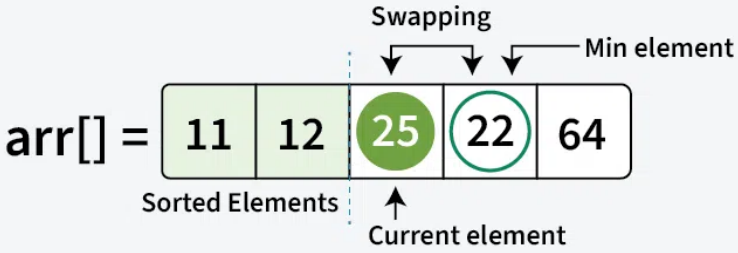
Selection Sort Algorithm

02 Step Move to the next element at index 1 (25). Find the smallest in unsorted subarray, and swap (12) with current element (25).



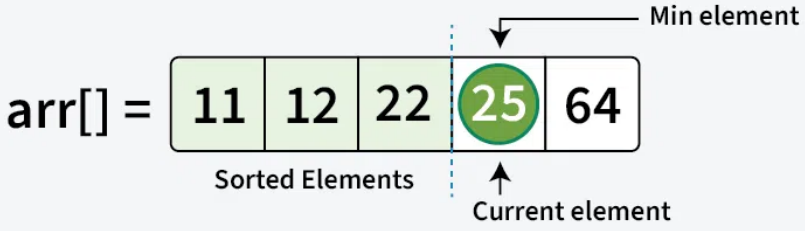
Selection Sort Algorithm

03 Step Move to element at index 2 (25). Find the minimum element from unsorted subarray, Swap (22) with current element (25).



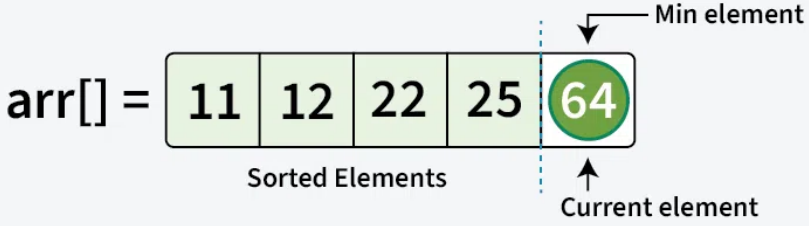
Selection Sort Algorithm

04 Step Move to element at index 3 (25), find the minimum from unsorted subarray and swap (25) with current element (25).



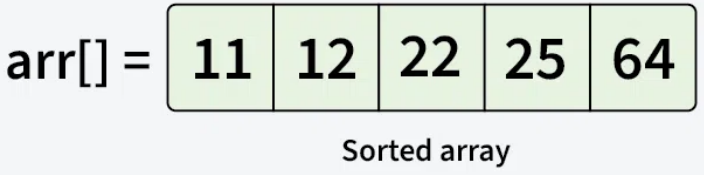
Selection Sort Algorithm

05 Step Move to element at index 4 (64), find the minimum from unsorted subarray and swap (64) with current element (64).



Selection Sort Algorithm

06 Step We get the sorted array at the end.

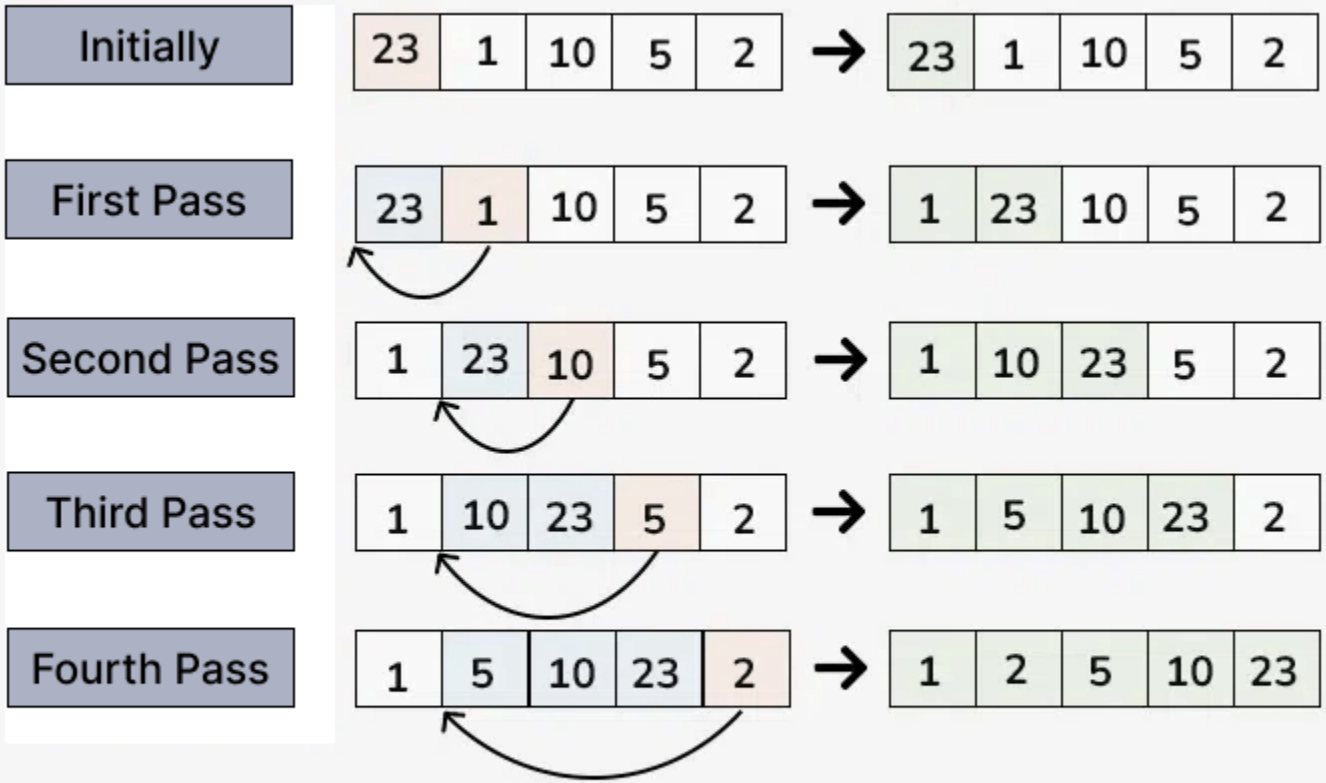


Selection Sort Algorithm

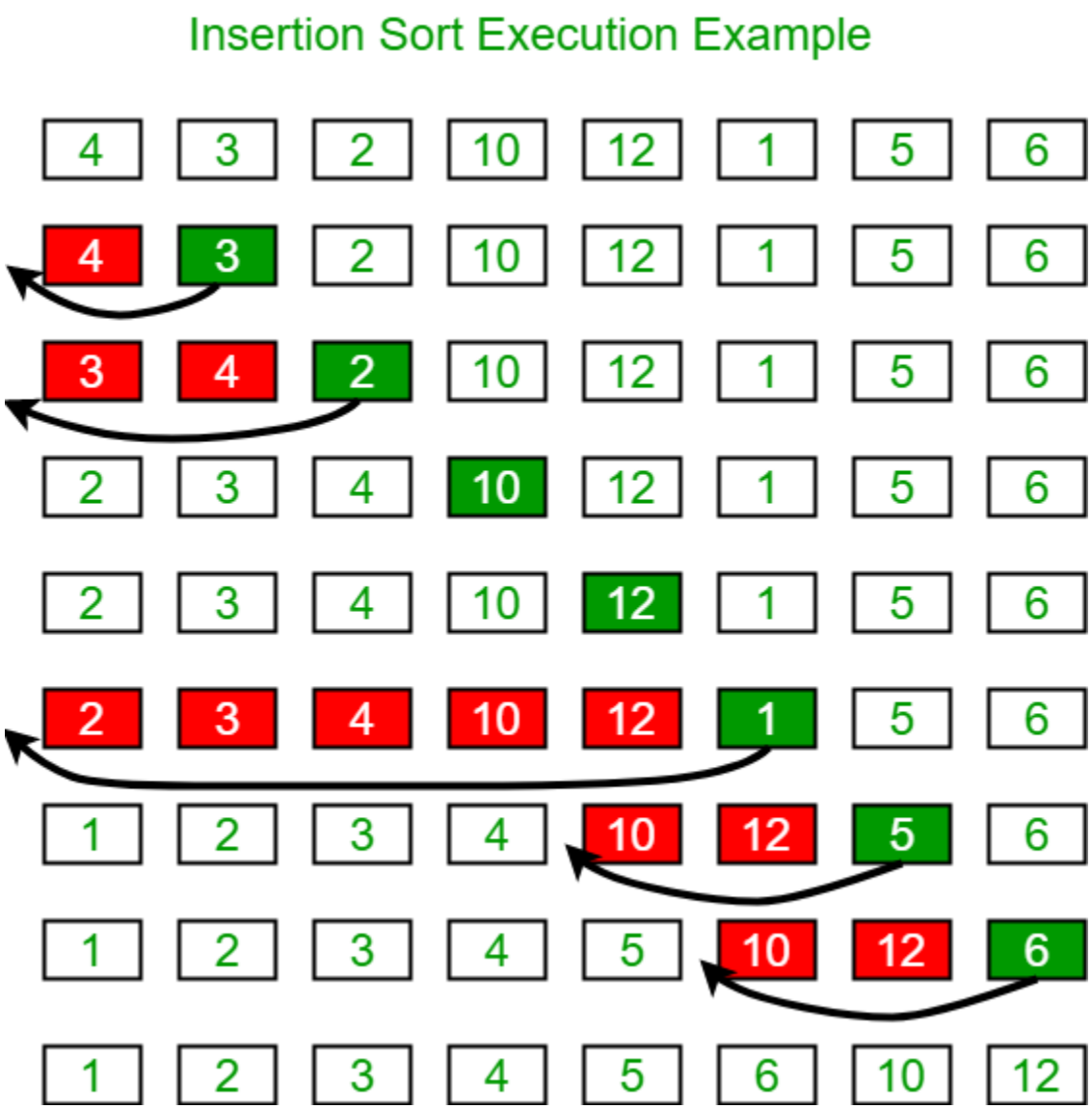
Insertion Sort

- Insertion sort works by iteratively inserting each element of an unsorted list into its correct position in a sorted portion of the list. It is like sorting playing cards in your hands. You split the cards into two groups: the sorted cards and the unsorted cards. Then, you pick a card from the unsorted group and put it in the right place in the sorted group.
  - We start with second element of the array as first element in the array is assumed to be sorted.
  - Compare second element with the first element and check if the second element is smaller then swap them.
  - Move to the third element and compare it with the first two elements and put at its correct position
  - Repeat until the entire array is sorted.
- Time complexity: $O(n^2)$, as there are two nested loops:
  - Outer loop to select each element in the unsorted group one by one, with O(n) complexity
  - Inner loop to insert that element into the sorted group, with O(n) complexity
- Insertion Sort | GeeksforGeeks
  - https://www.geeksforgeeks.org/insertion-sort-algorithm/
  - https://www.geeksforgeeks.org/time-and-space-complexity-of-insertion-sort-algorithm/
  - https://www.youtube.com/watch?v=OGzPmgsI-pQ

Insertion Sort Example 1

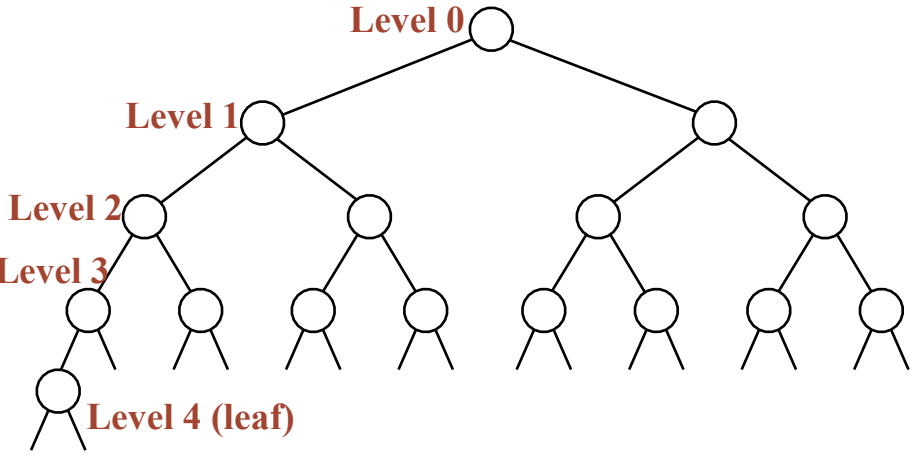


Insertion Sort Example 2

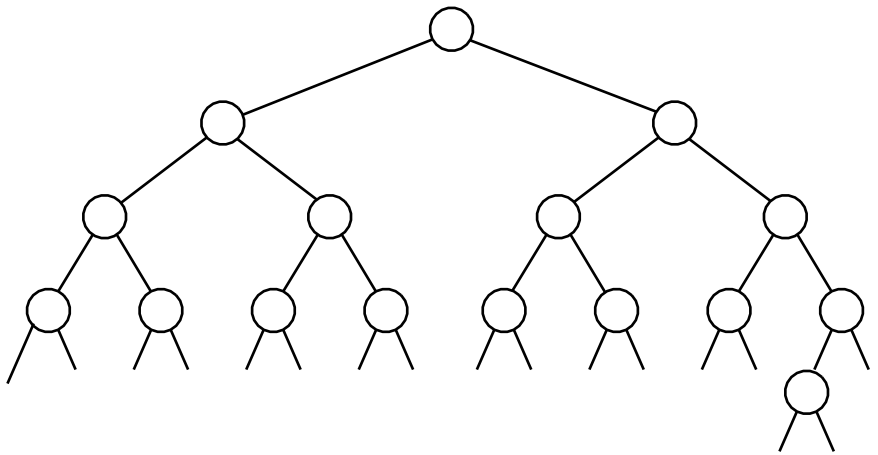


Heapsort: Binary Heap

- In a **heap** the largest (or smallest) element is always stored at the root, hence the name "heap". A heap is useful data structure when you need to remove the largest (or smallest) element. A common use of a heap is to implement a **priority queue** and **heapsort**.
- A **binary heap** is a complete binary tree which is an efficient data structure satisfies the heap ordering property.
  - In a *complete tree*, every level (except possibly the leaf level) is completely filled; the leaf level is filled from left to right.

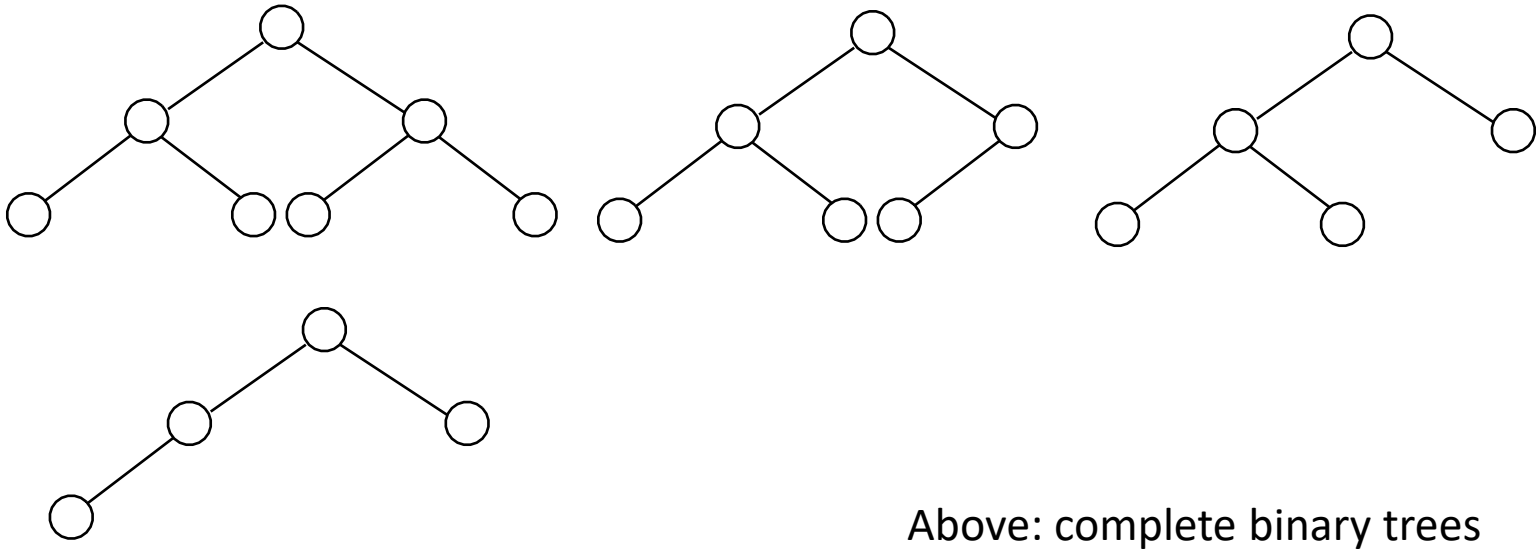


A complete binary tree with n = 16 nodes (height = 4)



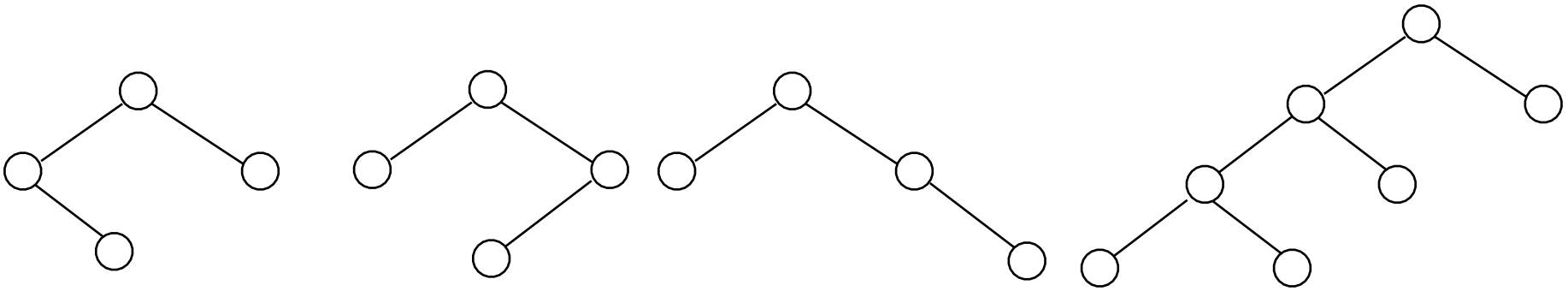
Not a complete binary tree (not filled from left to right)

Complete Binary Tree or Not?



Above: complete binary trees

Below: not complete binary trees

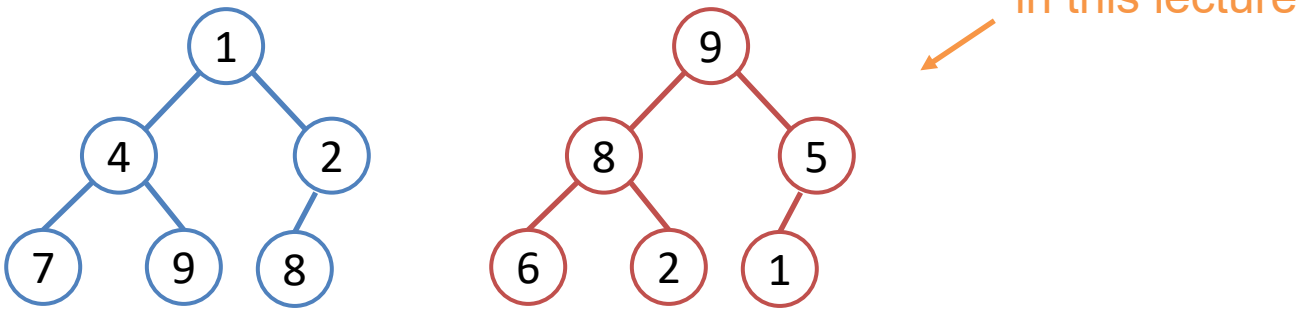


Leaf level is not filled from left to right.

Non-leaf level is not completely filled.

Heapsort: Binary Heap

- The heap ordering can be one of two types:
- The **min-heap property**: the value of each node is greater than or equal to the value of its parent, with the minimum-value element at the root.
- The **max-heap property**: the value of each node is less than or equal to the value of its parent, with the maximum-value element at the root.



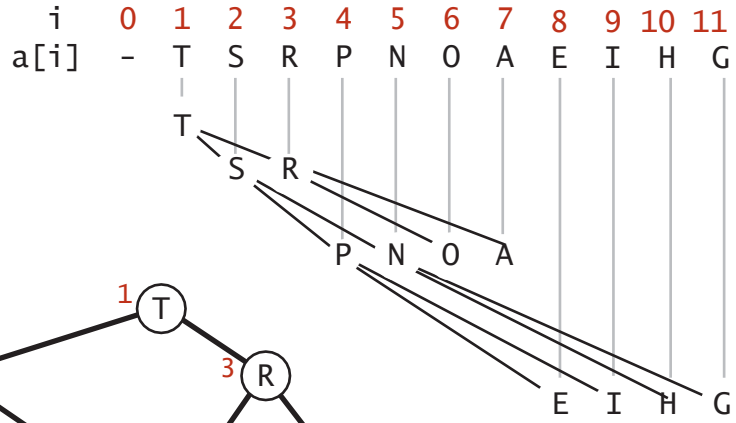
- A heap is not a sorted structure and can be regarded as partially ordered. There is no particular relationship among nodes on any given level, even among the siblings.
- Since a heap is a complete binary tree, it has a smallest possible height - a heap with n nodes has ***O(log n)*** height.

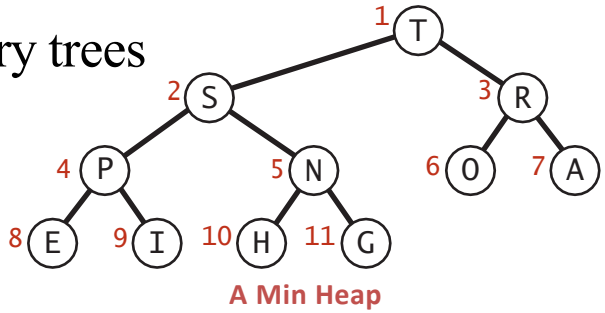
Max Heap Animations | Data Structure | Visual How
https://www.youtube.com/watch?v=uU0iWaVxMgc

Binary Heap: Array Representation

Array representation.

- Indices range in [1, n].
- Take nodes in level order.
- No explicit pointers needed!
- Only works for complete binary trees





A Min Heap

Proposition. Can use array indices to move through tree.

- Parent of node at index k is at floor(k/2).
  - Parent of node at 8 or 9 is at floor(8/2)= floor(9/2)=4.
- Children of node at index k are at 2k and 2k+1.
  - Children of node at 4 is at 2*4=8 and 2*4+1=9

Proposition. Largest key is a[1], which is root of binary tree.

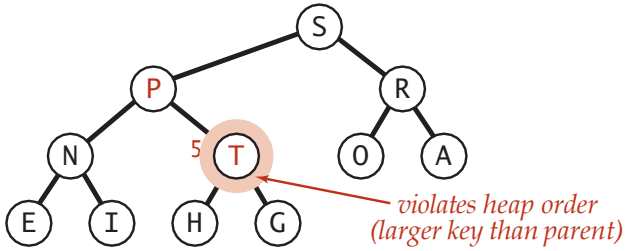
- Leaf nodes have indices [floor(n/2)+1, n]
  - [floor(11/2)+1, n]=[6, 11]
- Non-leaf nodes have indices [1, floor(n/2)]
  - [1, floor(11/2)]=[1, 5]

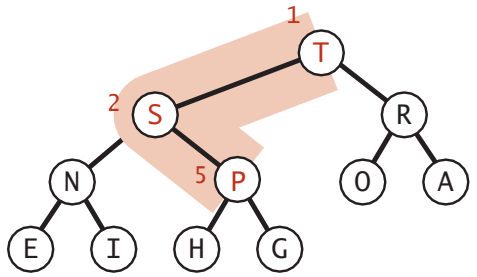
Binary Heap Operations: Promotion

- Scenario. A key becomes larger than its parent's key.
- To eliminate the violation:
- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

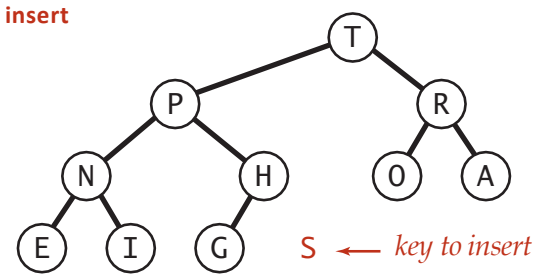
parent of node at k is at k/2

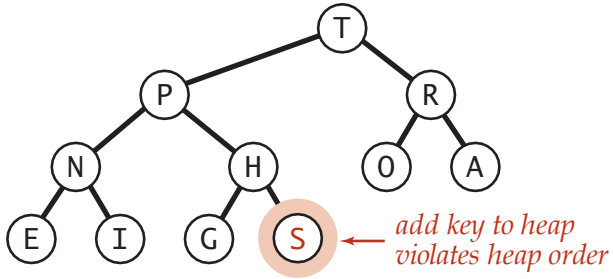


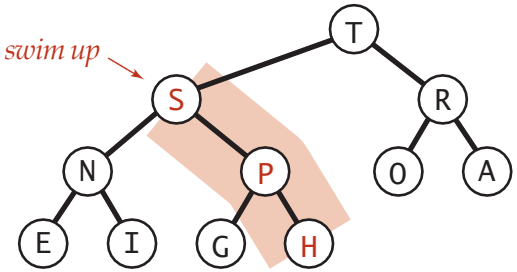


Binary Heap Operations: Insert

- Insert. Add node as leaf, then swim it up.
- Cost. O(log n) compares since tree height is O(log n).







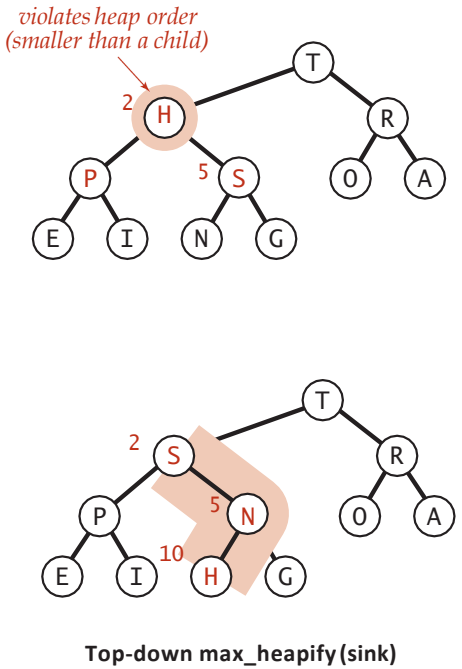

```
public void insert(Key x)
{
   pq[++n] = x;
   swim(n);
}
```

Binary Heap Operations: Demotion

- Scenario. A key becomes smaller than one (or both) of its children's.
- To eliminate the violation:
- Exchange key in parent with key in larger of the two children.
- Repeat until heap order restored.
- (Called max_heapify in video)

```
private void sink(int k, int n)
{
   while (2*k <= n)
   {
      int j = 2*k;
      if (j < n && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

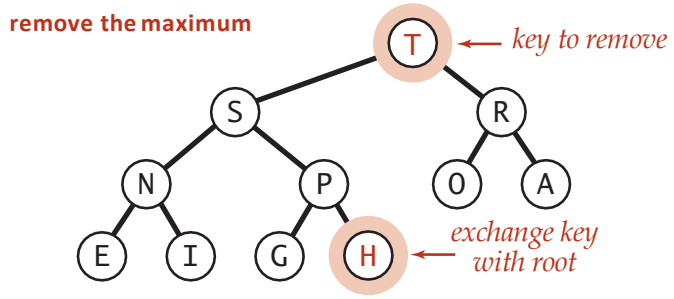
children of node at k are 2*k and 2*k+1

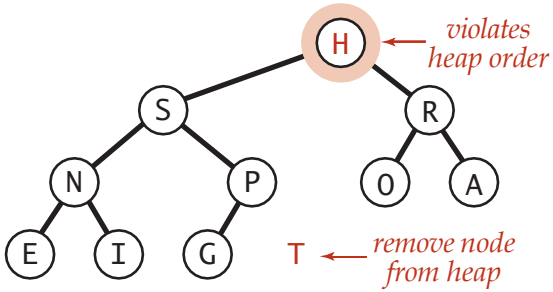


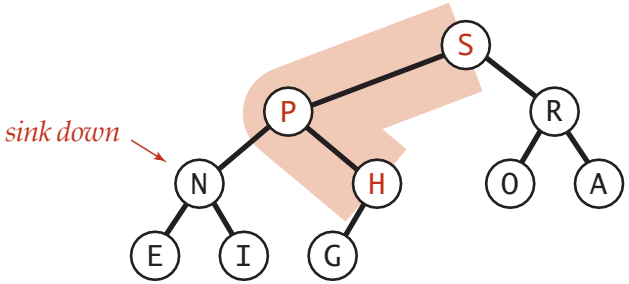
Top-down max_heapify (sink)

Binary Heap Operations: DeleteMax

- Delete max. Exchange root with rightmost leaf node (last element of the array), then sink it down.
- Cost. O(log n) compares.





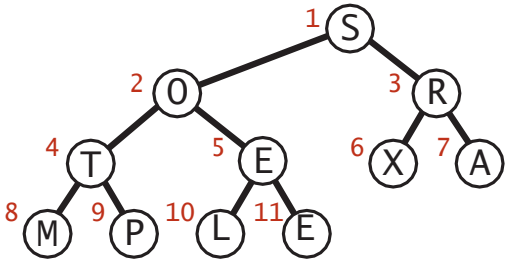



```
public Key delMax()
{
   Key max = pq[1];
   exch(1, n--);
   sink(1);
   pq[n+1] = null;
   return max;
}
```

Heapsort Algorithm

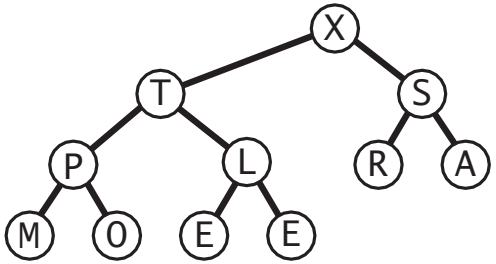
- Basic plan for in-place sort.
- View input array as a complete binary tree.
- Heap construction: build a max-heap with all n keys.
- Sortdown: repeatedly remove the maximum key.

keys in arbitrary order



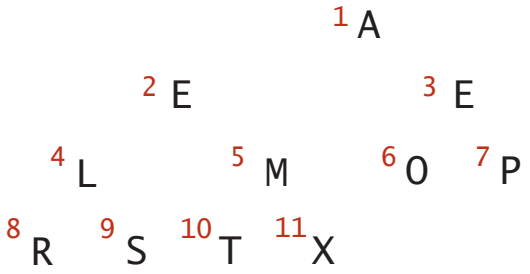
1	2	3	4	5	6	7	8	9	10	11
S	O	R	T	E	X	A	M	P	L	E

build max heap (in place)



1	2	3	4	5	6	7	8	9	10	11
X	T	S	P	L	R	A	M	O	E	E

sorted result (in place)

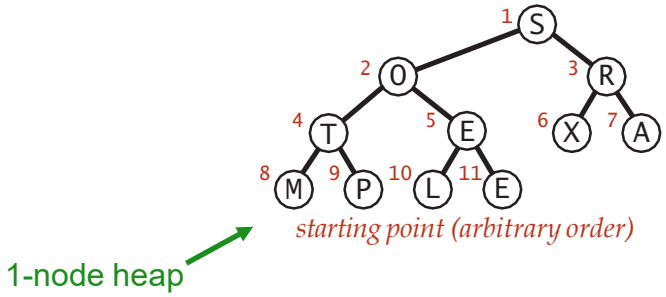


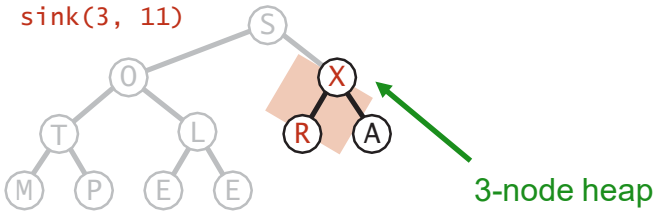
1	2	3	4	5	6	7	8	9	10	11
A	E	E	L	M	O	P	R	S	T	X

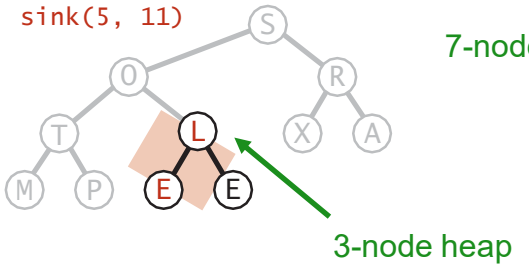
Heapsort: Heap Construction

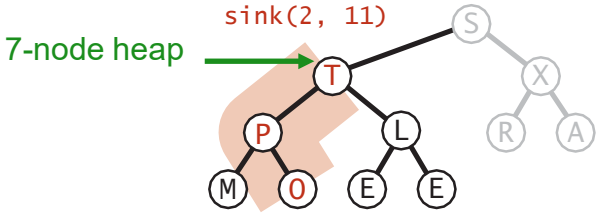
First pass. Build heap using bottom-up method.

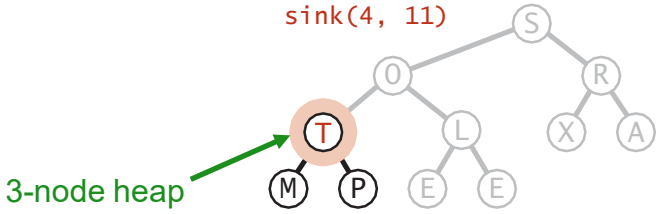
```
for (int k = floor(n/2); k >= 1; k--)
   //call sink(k) on all non-leaf
   nodes k from bottom up
   sink(k, n);
```

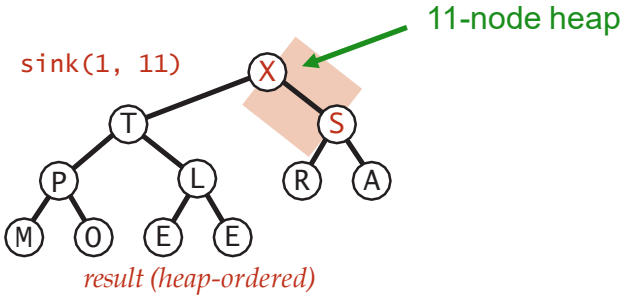










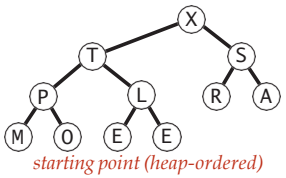


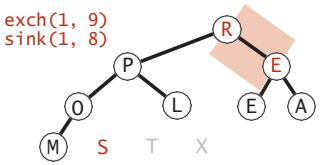
Heapsort: Sortdown

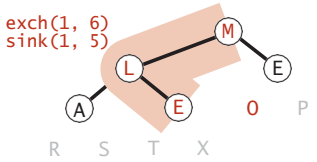
Second pass.

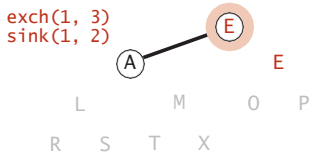
- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

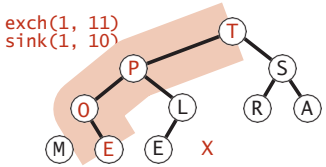
```
while (n > 1)
{
   exch(a, 1, n--);
   sink(a, 1, n);
}
```

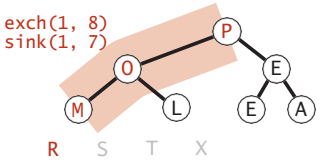


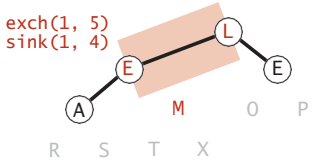


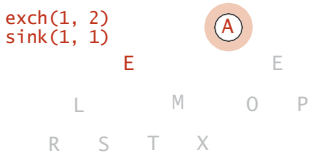


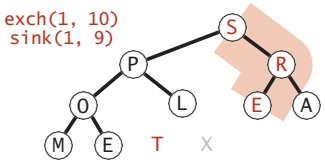


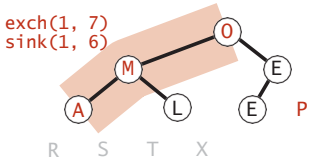


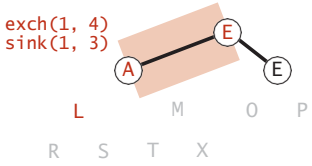


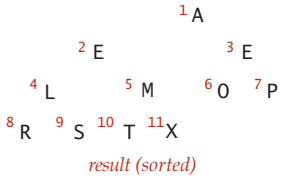












Heapsort: Java Implementation

```
public class Heap
{
   public static void sort(Comparable[] a)
   {
      int n = a.length;
      for (int k = n/2; k >= 1; k--)
          sink(a,  k,  n);
      while (n >   1)
      {
           exch(a,  1,  n);
           sink(a,  1,  --n);
      }
   }

   private static void sink(Comparable[] a, int k, int n)
   {  /* as before */  }

   private static boolean less(Comparable[] a, int i, int j)
   {  /* as before */  }

   private static void exch(Object[] a, int i, int j)
   {  /* as before */  }

}
```

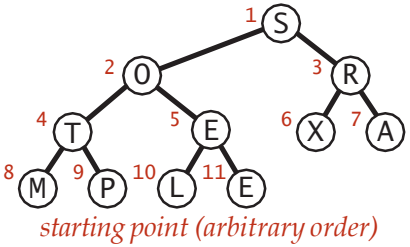
Integer arithmetic k=n/2 takes the floor of n/2

O(nlogn)

but make static (and pass arguments)

but convert from 1-based indexing to 0-base indexing

Heapsort: Trace



starting point (arbitrary order)

sink(k, N)

N	k	a[i] 0	1	2	3	4	5	6	7	8	9	10	11
initial	*values*		S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted	*result*		A	E	E	L	M	O	P	R	S	T	X

3-node heap

7-node heap

11-node heap

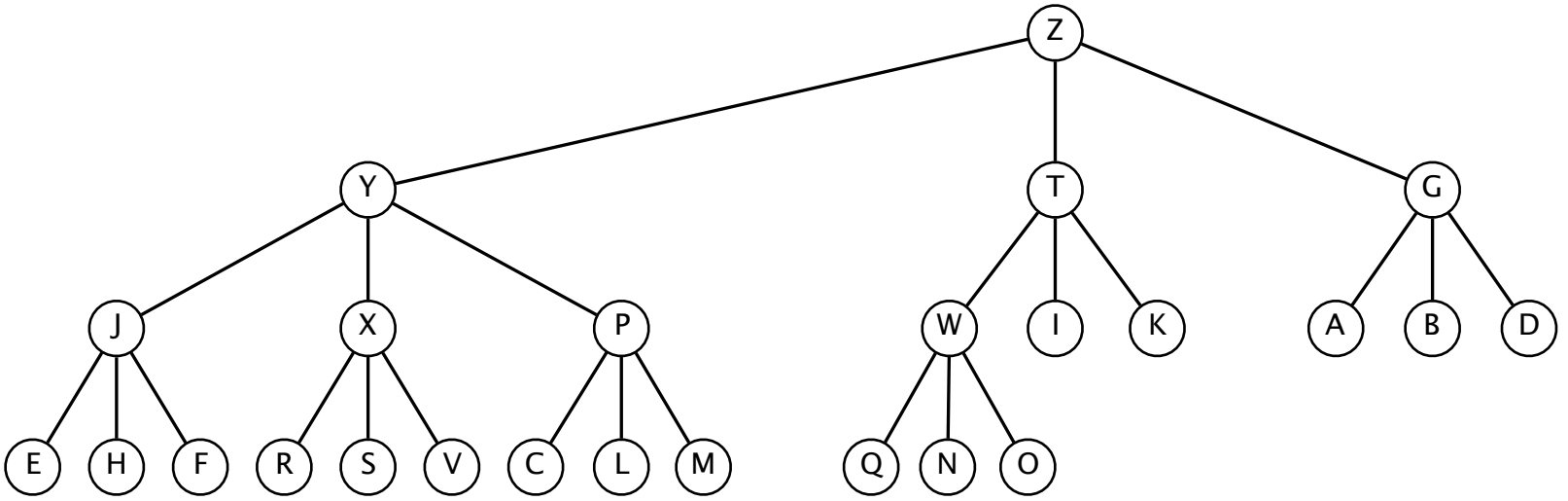
red: exchanged

black: compared

Heapsort trace (array contents just after each sink)

Binary Heap: Practical improvements

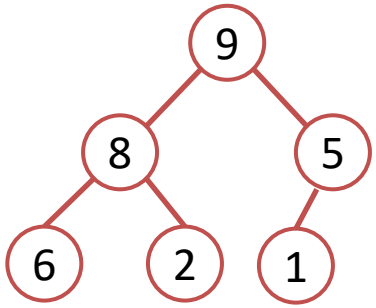
- Multiway heaps. Complete d-way tree.
- Parent's key no smaller than its children's keys.
- Fact. Height of complete d-way tree on n nodes is $O(\log_d n)$.



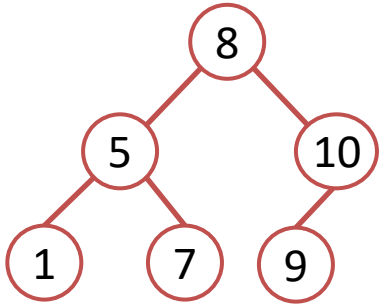
3-way heap

Binary Heap vs. Binary Search Tree

- Binary Heap is different from Binary Search Tree (BST, e.g., red-black tree)
- Binary Heap: the max-heap property
  - Value of each node is less than or equal to the value of its parent, with the maximum-value element at the root.
  - A heap is not a sorted structure and can be regarded as partially ordered.
- BST: Ordered, or sorted, binary trees
  - Items to the left of a given node are smaller.
  - Items to the right of a given node are larger.
- Both structures offer O(log n) time complexity for certain operations, they are used in different scenarios.
  - Heapsort is used for efficient sorting and simple priority queue implementations
  - BST can also be used for sorting, by insertions followed by in-order traversal, with O(n log(n)) average-case complexity.
    - Red-black trees are for maintaining ordered data with frequent updates and searches.



Binary Heap



Binary Search Tree
In-order traversal gives sorted list [1,5,7,8,9,10]

Quicksort

- QuickSort is based on the principle of divide and conquer: breaking down the problem into smaller sub-problems.
- **Choose a Pivot**: Select an element from the array as the pivot. The choice of pivot can vary (e.g., first element, last element, random element, or median).
- **Partition the Array**:
- 1. The pivot is compared with each element in the array.
- 2. Elements smaller than the pivot are moved to its left.
- 3. Elements larger than the pivot are moved to its right.
- 4. The pivot is placed in its final sorted position.
- **Recursively Call**: Recursively apply the same process to the two partitioned sub-arrays (left and right of the pivot).
- **Base Case**: The recursion stops when there is only one element left in the sub-array, as a single element is already sorted.
- QuickSort | geeksforgeeks
  - https://www.geeksforgeeks.org/quick-sort-algorithm/
  - QUICK SORT | Sorting Algorithms | DSA | GeeksforGeeks
  - https://www.youtube.com/watch?v=PgBzjlCcFvc&t=80s
  - Partition Function of Quick Sort | GeeksforGeeks https://www.youtube.com/watch?v=OwR53k9DZ9c

Quicksort: Selection of Pivot

- The pivot is an element chosen from the array that serves as a reference point for partitioning the array into two subarrays. There are several strategies for selecting the pivot:
  - ###First or Last Element as Pivot. Simple to implement but can lead to poor performance if the array is already sorted or nearly sorted.
  - ### Random Element as Pivot. This can help avoid worst-case scenarios and provide more consistent performance across different input distributions.
  - ### Median-of-Three. This method selects the median of the first, middle, and last elements of the array as the pivot. It provides a good balance between simplicity and performance.

Quicksort Time Complexity

Worst case. Number of compares is quadratic.

- $n + (n - 1) + (n - 2) + \ldots + 1 \rightarrow O(n^2)$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39\, n \log n$.

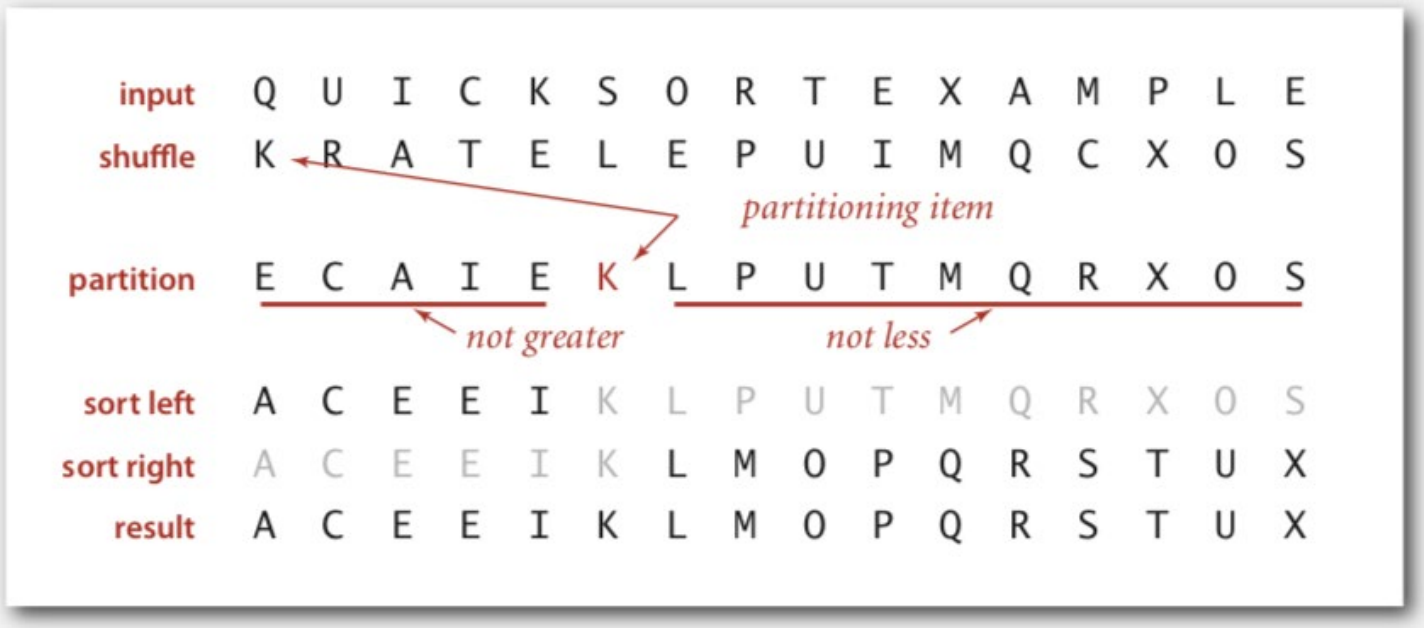
- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Quicksort Example 1

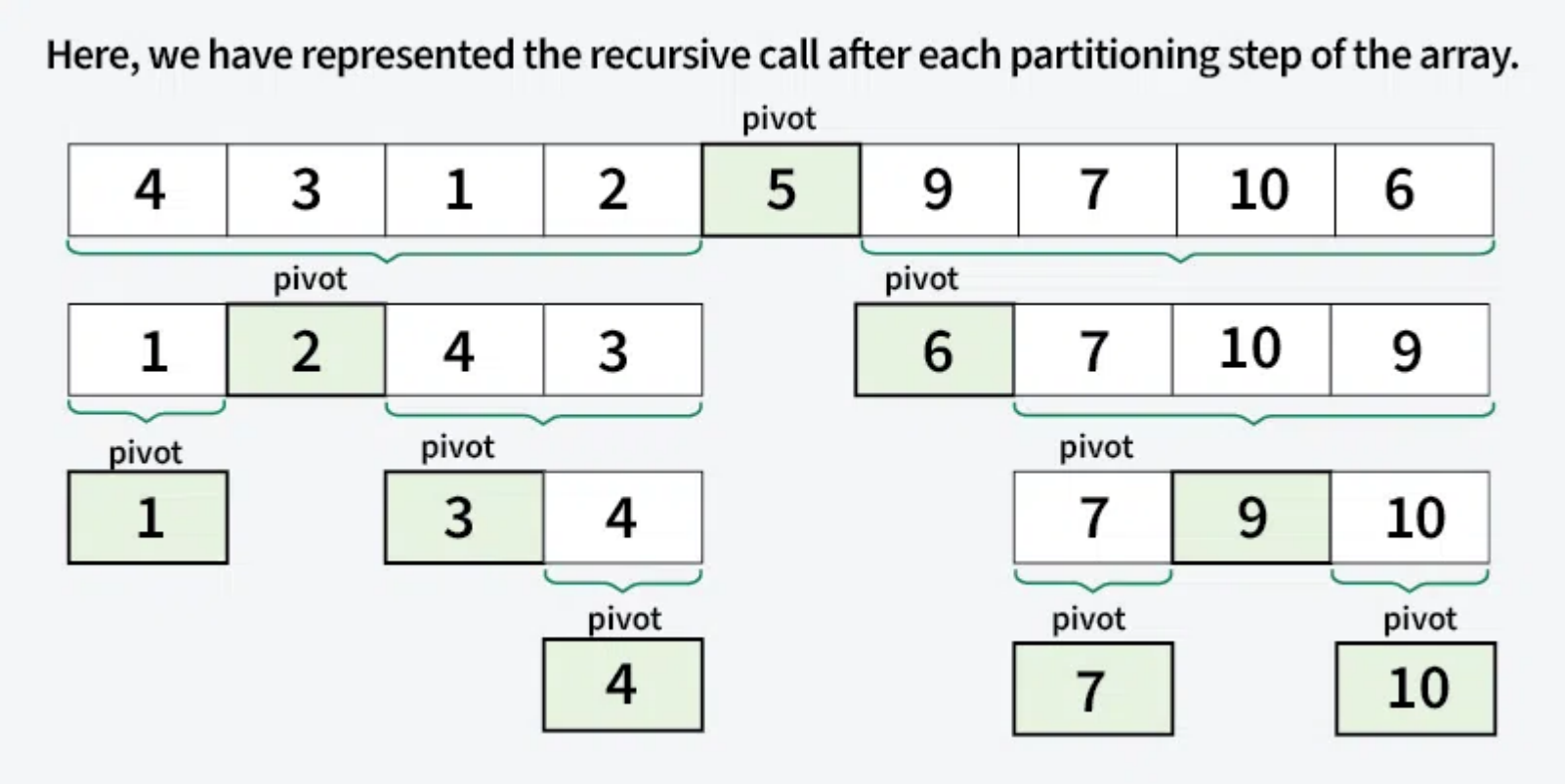
1. Shuffle the array.
2. Partition so that, for some pivot j
   - entry a[j] is in place
   - no larger entry to the left of j
   - no smaller entry to the right of j
3. Sort each piece recursively.



Choose the first element as pivot

Quicksort Example 2

- Input array [4, 3, 9, 7, 1, 2, 10, 6, 5]



Choose the last element as pivot

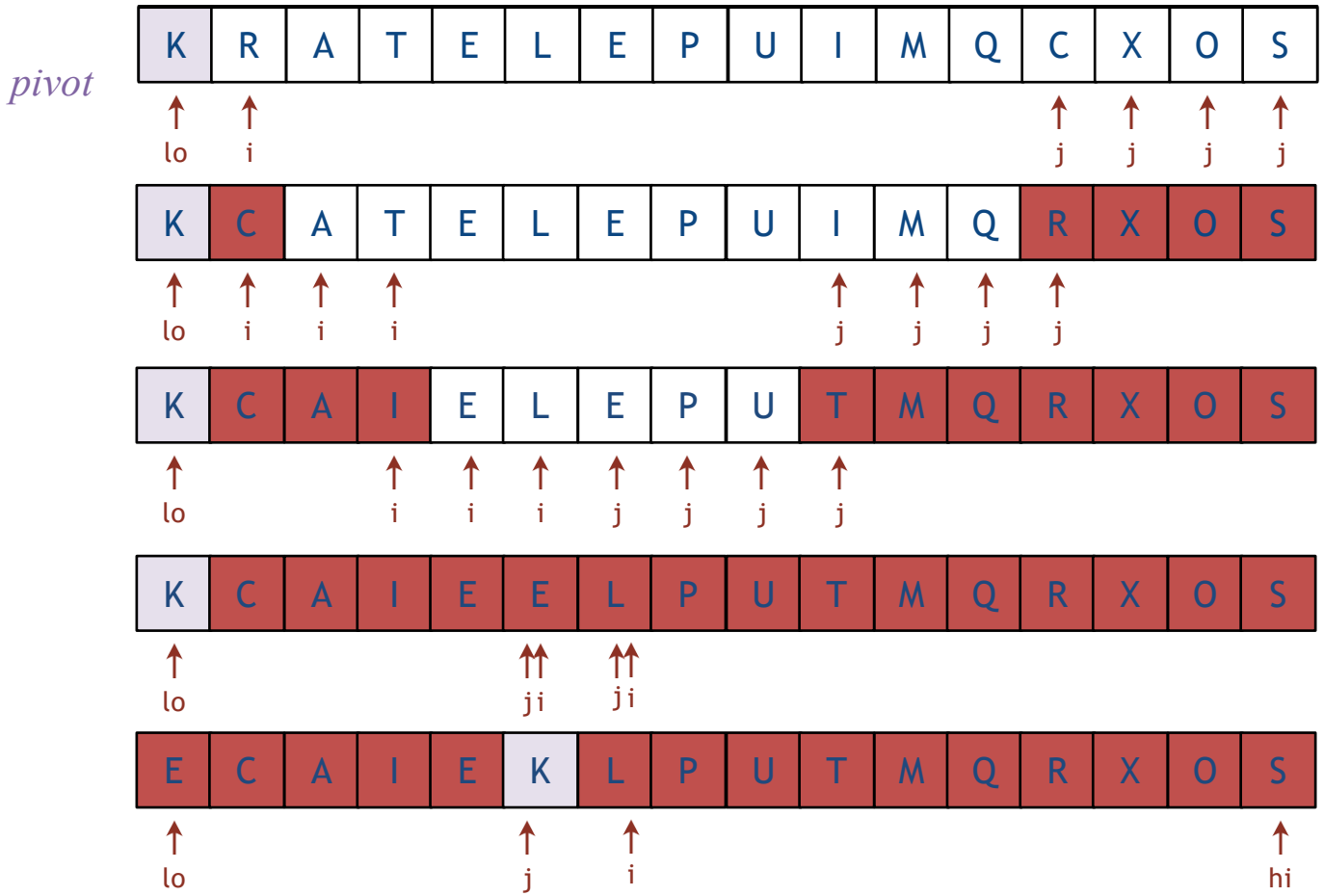
Partition Operation

Repeat until i and j pointers cross.

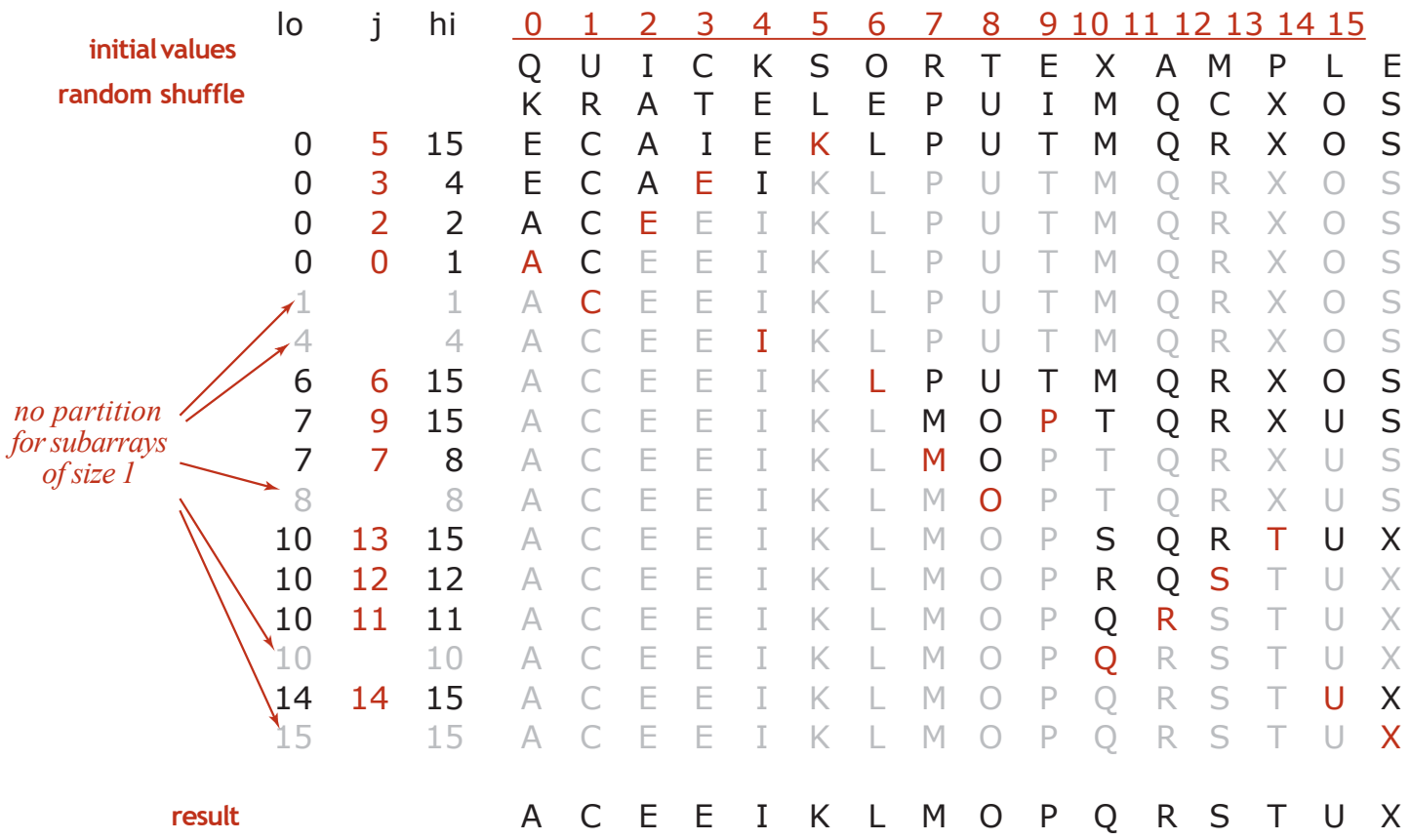
- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].

When pointers cross.

- Exchange a[lo] with a[j].



Quicksort: Trace



Quicksort trace (array contents after each partition)

Partition Operation: Java Implementation

```
private static int partition(Comparable[] a, int lo, int hi)
{
    int i = lo, j = hi+1;
    while (true)
    {
        while (less(a[++i], a[lo]))
        if (i == hi) break;

        while (less(a[lo], a[--j]))
        if (j == lo) break;

        if (i >= j) break;
        exch(a, i, j);
    }

    exch(a, lo, j);
    return j;
}
```

find item on left to swap

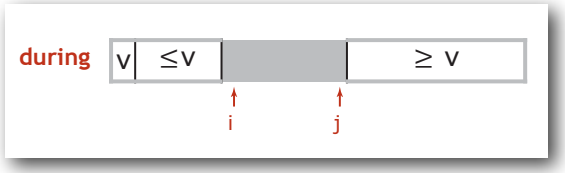
find item on right to swap

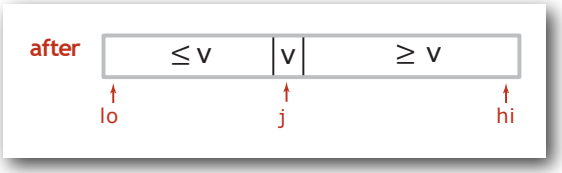
check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place





Quicksort: Java Implementation

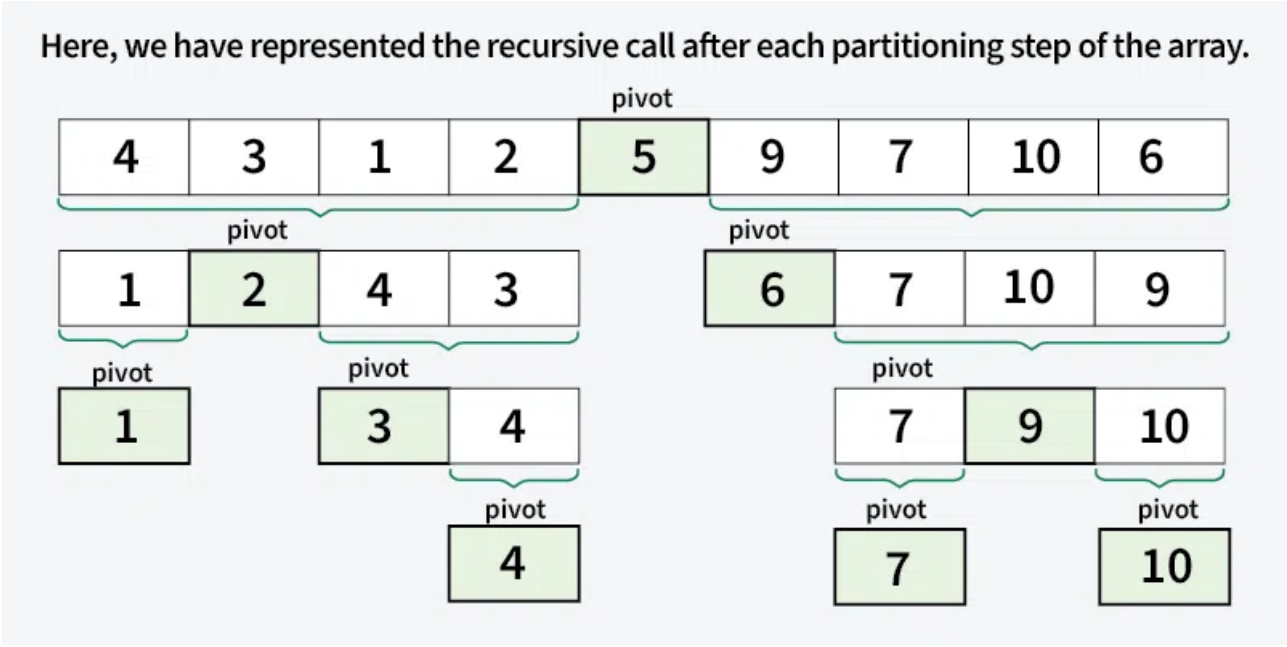
```
public class Quick
{
   private static int     partition(Comparable[]  a,  int   lo,  int   hi)
      {/* see previous slide /     } *

    public static void sort(Comparable[] a)
    {
        StdRandom.shuffle(a);
        sort(a, 0, a.length - 1);
    }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
   }
}
```

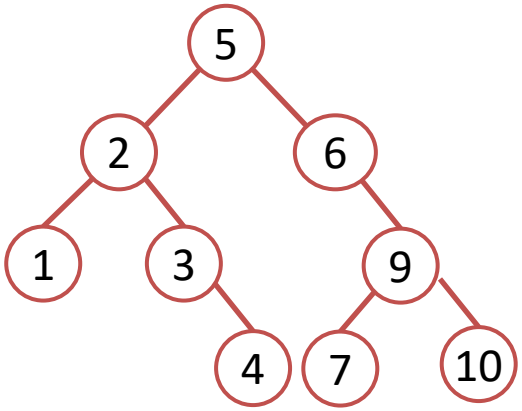
Shuffle needed to improve performance

Pivot selection

Quicksort is Equivalent to Sorting by BST

- Key idea: compareTo calls are same for BST insert and Quicksort.
  - Every number gets compared to 5; 1, 3, 4 get compared to only 2.
- Recall: Insertion into a BST has average-case complexity O(N log N).





Binary Search Tree
In-order traversal gives sorted list
[1,2,3,4,5,6,7,9,10]

Quicksort: Best-case Analysis

Best case. Number of compares is ~ N log N.

- When the list is randomly shuffled:
  - If we happen to always pick the median element as the pivot, then Quicksort has the best-case complexity of O(n log n). This corresponds to a balanced BST.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: Worst-case Analysis

Worst case. Number of compares is $O(N^2)$.

- When the list is already sorted:
  - If we always pick the first element as the pivot (shown in the right), then Quicksort has the worst-case complexity of $O(n^2)$. This corresponds to a very unbalanced BST.
  - If we always pick the middle element as the pivot, then Quicksort has the best-case complexity of O(n log n). This corresponds to a balanced BST.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: Practical Improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

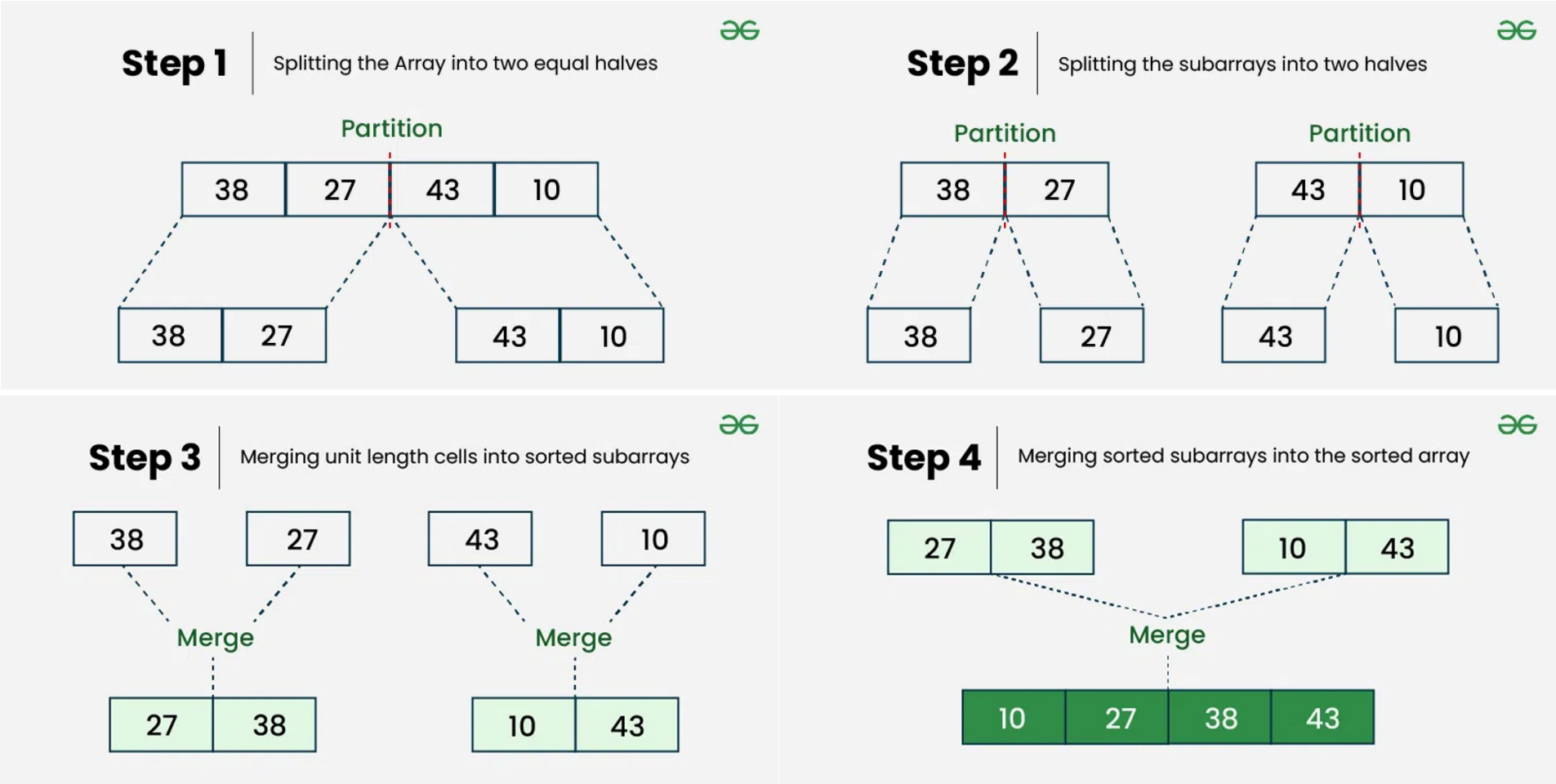
```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Mergesort

- Mergesort is based on the principle of divide and conquer: breaking down the problem into smaller sub-problems.
  - Divide array into two halves.
  - Recursively sort each half.
  - Merge two halves.

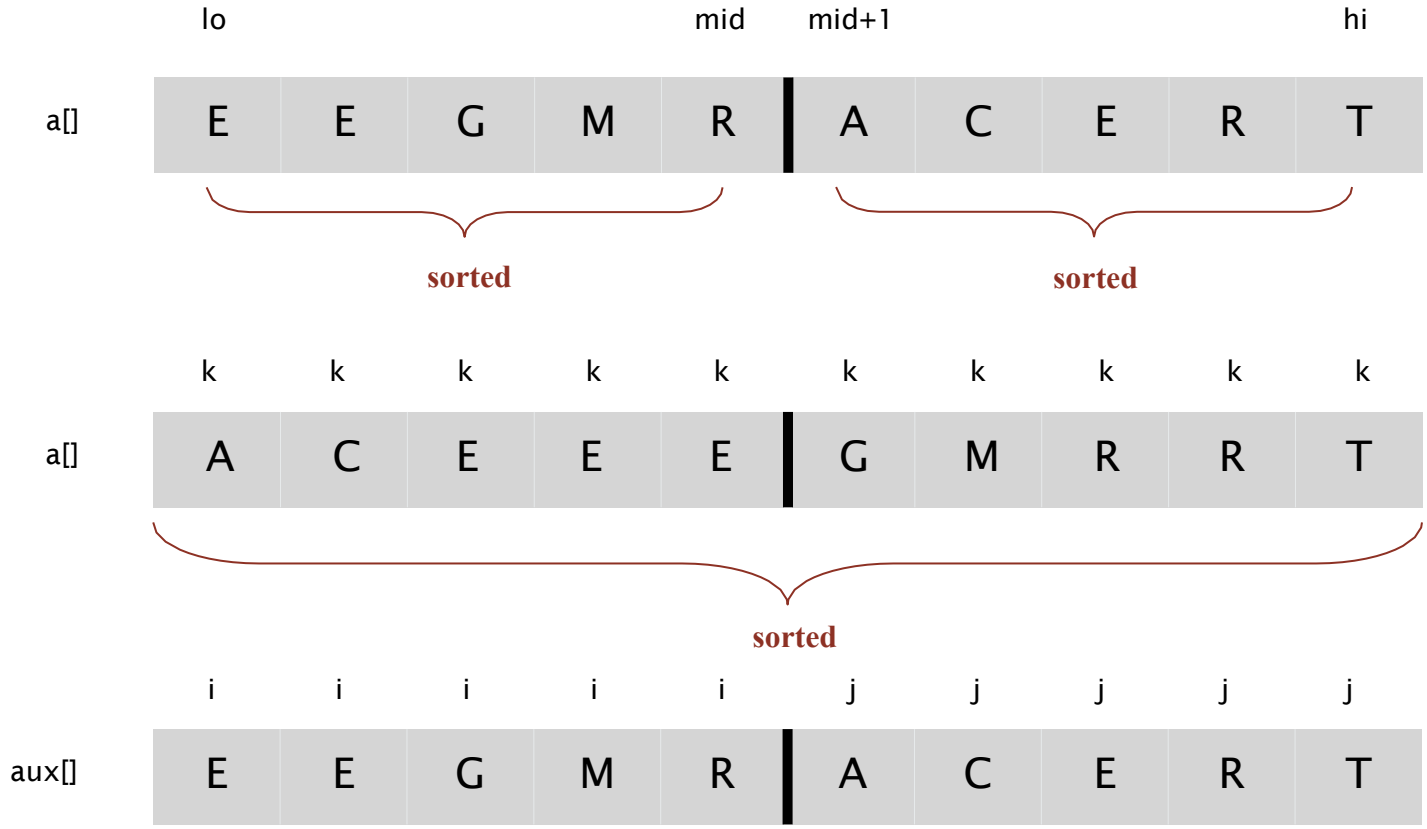
Mergesort overview

Mergesort Example 1



Merge Operation

Goal. Given two sorted subarrays a[lo] to a[mid] and a[mid+1] to a[hi], obtain sorted array a[lo] to a[hi], by selecting the smaller of the smallest elements in each subarray and put it into the sorted array a[].

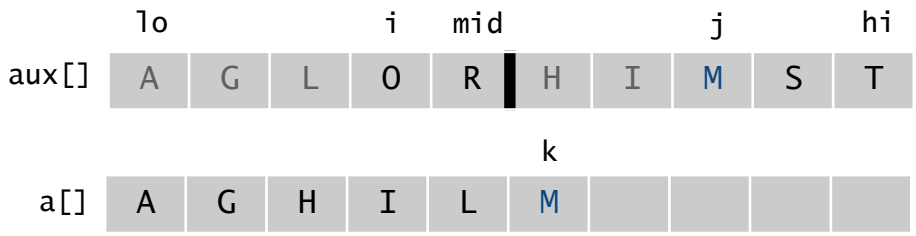


one subarray exhausted, take from other

Merge Sort Algorithm: A Step-by-Step Visualization
https://www.youtube.com/watch?v=ho05egqcPl4

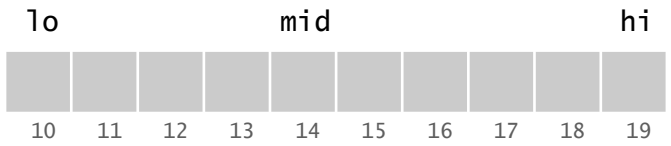
Merge Operation: Java Implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   assert isSorted(a, lo, mid);    // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);  // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)                                          copy
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)                                          merge
   {
      if      (i > mid)                 a[k] = aux[j++];
      else if (j > hi)                  a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                              a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);     // postcondition: a[lo..hi] sorted
}
```

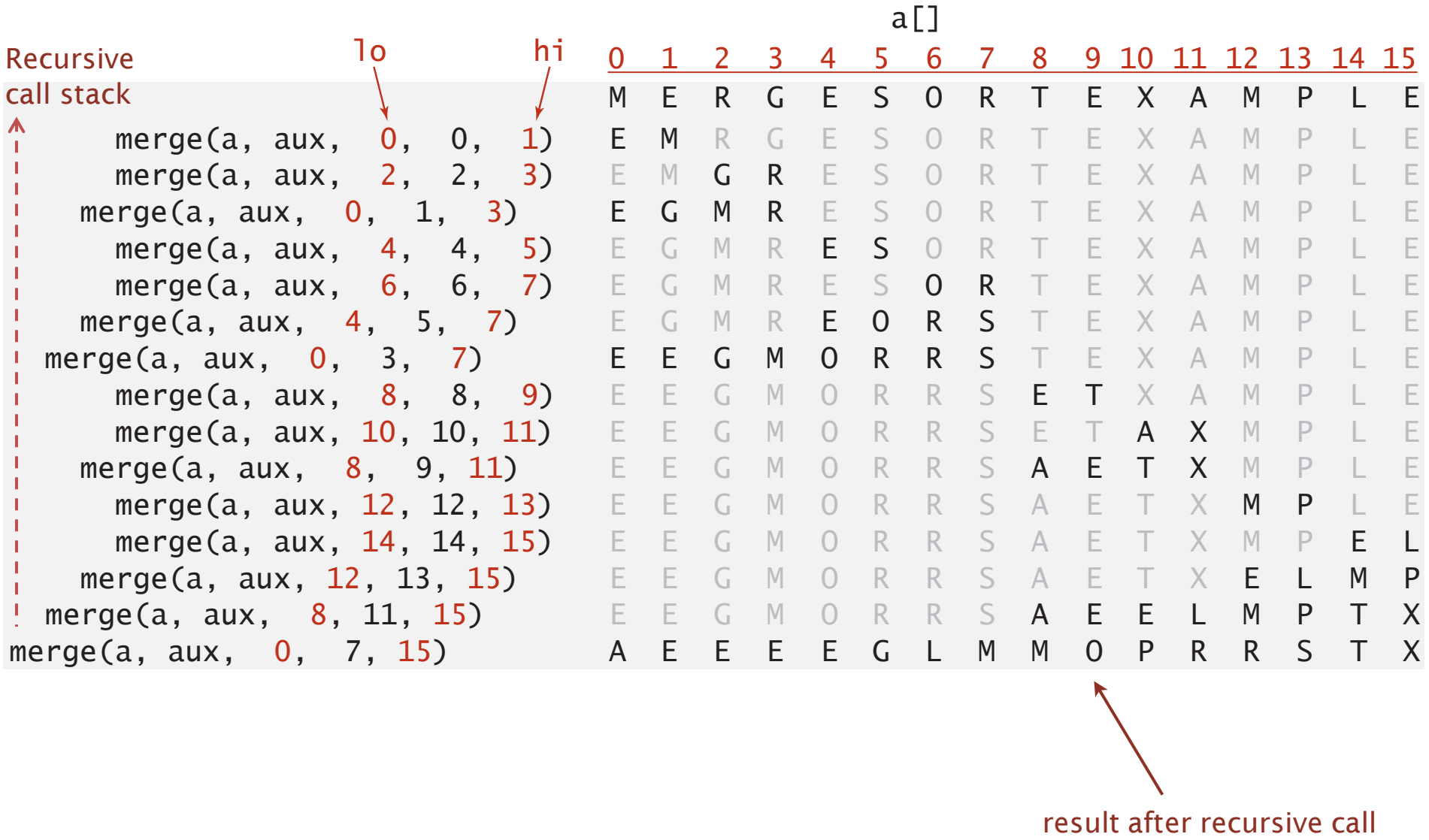


Mergesort: Java implementation

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



Mergesort: Example 2 Trace



Mergesort: Practical Improvements

Use insertion sort for small subarrays.

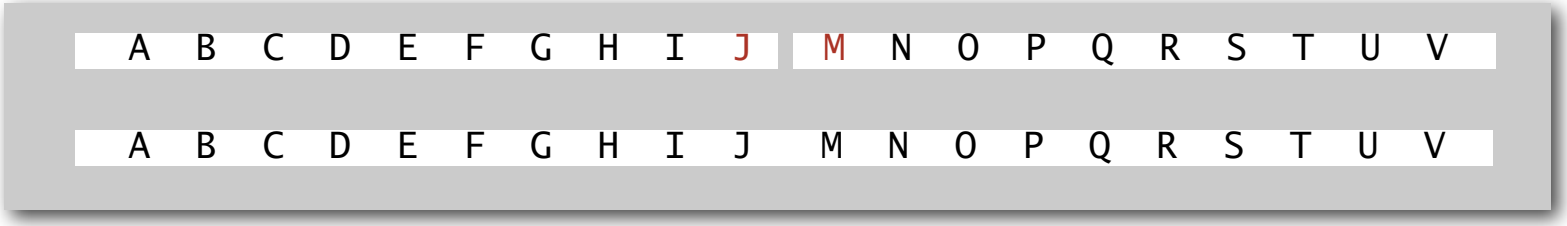
- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: Practical Improvements

Stop if already sorted.

- Is biggest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

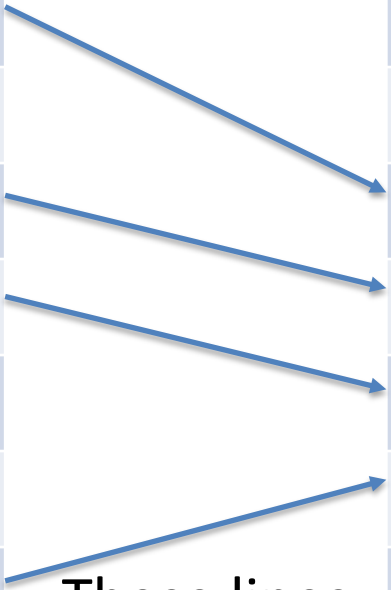
Stable Sorting Algorithm

- A stable sorting algorithm is one that maintains the relative order of elements with equal keys in the sorted output as they appeared in the input.
- Stability is important when multiple sorting operations are performed on data with multiple keys. For example, if you first sort a list of students by name and then by grade, a stable sort will ensure that students with the same grade remain sorted by name. This characteristic is crucial in scenarios where secondary attributes need to be preserved after sorting by primary attributes.
- Stable Sorting Algorithms include: Bubble Sort, Insertion Sort, Merge Sort, Radix Sort (next lecture)

Stable Sorting Example

Name	Grade
Bas	60
Frank	80
Jana	60
Jouni	60
Lara	20
Nick	80
Rose	60
Sam	40

Sort by Name
(primary attribute)



Name	Grade
Lara	20
Sam	40
Bas	60
Jana	60
Jouni	60
Rose	60
Frank	80
Nick	80

Then sort by Grade
(secondary attribute)

Summary

	inplace?	stable?	complexity (worst-case)	remarks
Bubble	✔	✔	$O(n^2)$	
Selection	✔		$O(n^2)$	
Insertion	✔	✔	$O(n^2)$	
J Heap	✔		O(n log n)	O(n log n) guarantee
Quick	✔		$O(n^2)$	O(n log n) probabilistic guarantee; fastest in practice
Merge		✔	O(n log n)	O(n log n) guarantee;

Video Tutorials

- Heap Sort
  - Heaps // Michael Sambol
    - https://www.youtube.com/playlist?list=PL9xmBV_5YoZNsyqgPW-DNwUeT8F8uhWc6
  - Binary Min/Max Heap
    - https://www.youtube.com/playlist?list=PLvTjg4siRgU197GA1yFNRWUgsPZnvjuyL
  - Visual How (min heap, max heap)
    - https://www.youtube.com/@visualhow/videos
  - HEAP SORT | Sorting Algorithms | DSA | GeeksforGeeks
    - https://www.youtube.com/watch?v=MtQL_ll5KhQ
  - 2.6.3 Heap - Heap Sort - Heapify - Priority Queues (recommended)
    - https://www.youtube.com/watch?v=HqPJF2L5h9U&list=PLDN4rrl48XKpZkf03iYFl-O29szjTrs_O&index=32

Video Tutorials

- Quick Sort
  - Quick sort in 4 minutes (recommended)
    - https://www.youtube.com/watch?v=Hoixgm4-P4M
  - Quicksort Algorithm: A Step-by-Step Visualization
    - https://www.youtube.com/watch?v=bZkzH5x0SKU
  - Visualization of Quick sort (HD)
    - https://www.youtube.com/watch?v=aXXWXz5rF64
- Merge Sort
  - Merge sort in 3 minutes
    - https://www.youtube.com/watch?v=4VqmGXwpLqc
  - Merge Sort Algorithm: A Step-by-Step Visualization (recommended)
    - https://www.youtube.com/watch?v=ho05egqcPl4
  - Merge Sort Animations | Data Structure | Visual How
    - https://www.youtube.com/watch?v=spVhtO_IcGg

Video Tutorials

- Sort Algos // Michael Sambol Michael Sambol
  - https://www.youtube.com/playlist?list=PL9xmBV_5YoZOZSbGAXAPIq1BeUf4j20pl
  - Merge Sort, Quick Sort, Bubble Sort, Insertion Sort, Selection Sort, Heap Sort
- 10 Sorting Algorithms Easily Explained
  - https://www.youtube.com/watch?v=rbbTd-gkajw
  - Bubble Sort, Selection Sort, Insertion Sort, Merge Sort, Quick Sort, Heap Sort, Counting Sort, Shell Sort, Tim Sort, Radix Sort